MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to the Final Exam

Problem 1 For one implication, assume $A \subseteq \mathscr{P}B$. Then by Problem 1.4, $\bigcup A \subseteq \bigcup \mathscr{P}B$, and by Problem 2.6(a), $\bigcup \mathscr{P}B = B$, thus $\bigcup A \subseteq B$. For the converse, assume $\bigcup A \subseteq B$. We showed in class that $X \subseteq Y \Rightarrow \mathscr{P}X \subseteq \mathscr{P}Y$, and thus $\mathscr{P}\bigcup A \subseteq \mathscr{P}B$. By Problem 2.6(b), $A \subseteq \mathscr{P}\bigcup A$, and thus $A \subseteq \mathscr{P}B$.

Problem 2 First suppose f is one-to-one, and let $X, Y \subseteq A$. We know that $f[\![X \cap Y]\!] \subseteq f[\![X]\!] \cap f[\![Y]\!]$ by Theorem 3K(b). For the converse inclusion, suppose $z \in f[\![X]\!] \cap f[\![Y]\!]$. Then $z = f(x)$ for some $x \in X$ and $z = f(y)$ for some $y \in Y$. But since f is one-to-one, $x = y$, and thus $x \in X \cap Y$ and $z \in f[\![X \cap Y]\!]$. It follows that $f[\![X \cap Y]\!] = f[\![X]\!] \cap f[\![Y]\!]$.

Conversely, suppose that for all $X, Y \subseteq A$, $f[\![X \cap Y]\!] = f[\![X]\!] \cap f[\![Y]\!]$. To show that f is one-to-one, assume $f(x) = f(y)$ for $x \neq y \in A$. Let $X = \{x\}$ and $Y = \{y\}$. Then $f[\![X]\!] \cap f[\![Y]\!] = \{f(x)\}$, whereas $f[\![X \cap Y]\!] = f[\![\emptyset]\!] = \emptyset$, a contradiction. Thus, f is one-to-one.

Problem 3 Suppose R and R^{-1} are both well-orderings on some set S. Let α be the ordinal number of $\langle S, R\rangle$. If S is infinite, then $\alpha \notin \omega$, hence $\omega \in \alpha$ by trichotomy, hence $\omega \subseteq \alpha$. Therefore α, and thus $\langle S, R\rangle$, has an infinite increasing chain. But this amounts to an infinite decreasing chain in $\langle S, R^{-1}\rangle$, contradicting the fact that R^{-1} is a well-order on S.

Problem 4 Recall that an ordinal is a transitive set of transitive sets. If A is a non-empty set of ordinals, then $\bigcap A$ is transitive, and so are its elements. So $\bigcap A$ is an ordinal. If B is any set of ordinals, then $\bigcup B$ is transitive, and so are its elements, so $\bigcup B$ is an ordinal. So the answer to the first two questions is yes. However, the set-theoretic difference of two ordinals is not usually an ordinal: For instance, $2 - 1 = \{\emptyset, \{\emptyset\}\} - \{\emptyset\} = \{\{\emptyset\}\}$, which is not an ordinal.

Problem 5 Let $\mathscr{A}$ be the set of antichains in P. We claim that $\mathscr{A}$ is closed under unions of chains: Consider any chain $\mathscr{C} \subseteq \mathscr{A}$. We must show that $\bigcup \mathscr{C}$ is an antichain. So consider any $x, y \in \bigcup \mathscr{C}$. Then $x \in C_1$ and $y \in C_2$ for some $C_1, C_2 \in \mathscr{C}$. Since $\mathscr{C}$ is a chain, we have $C_1, C_2 \subseteq C_i$ for some $i \in \{1, 2\}$. So $x, y \in C_i$. But C_i is an antichain, and thus $x \not< y$. Since $x, y \in \bigcup \mathscr{C}$ were arbitrary, it follows that $\bigcup \mathscr{C}$ is an antichain, and thus $\bigcup \mathscr{C} \in \mathscr{A}$. Thus, $\mathscr{A}$ is closed under unions of chains and has a maximal element by Zorn's lemma. This element is a maximal antichain in P.

Problem 6 Recall that $\operatorname{card}\mathbb{R} = 2^{\aleph_0}$ and $\operatorname{card}\omega = \aleph_0$. Thus we have

$$\operatorname{card}({}^{\omega}\mathbb{R}) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \operatorname{card}\mathbb{R}.$$

Problem 7 Let S be a set of cardinals. We know that $\bigcup S$ is an ordinal by Problem 4b. It remains to be shown that $\bigcup S$ is a cardinal, i.e., that for all $\beta \in \bigcup S$, $\beta \not\approx \bigcup S$. So take $\beta \in \bigcup S$. By definition of union, there is $\kappa \in S$ with $\beta \in \kappa$. Because β, κ, and $\bigcup S$ are all ordinals, we have $\beta \subseteq \kappa \subseteq \bigcup S$, and thus $\beta \preccurlyeq \kappa \preccurlyeq \bigcup S$. If $\beta \approx \bigcup S$, then we would have $\beta \approx \kappa$ by the Schröder-Bernstein Theorem, but this can't be since $\beta \in \kappa$ and κ is a cardinal. It follows that $\beta \not\approx \bigcup S$, as desired. Thus, $\bigcup S$ is not equinumerous to any smaller ordinal, and hence it is a cardinal.

Problem 8 An ordinal is, by definition, a transitive set of transitive sets. We also know that ordinals are well-ordered by $\in$, and thus in particular they are $\in$-connected. So the left-to-right implication is trivial.

Conversely, let A be a transitive set that is $\in$-connected. To show that A is an ordinal, we must show that each $x \in A$ is transitive. So consider $z \in y \in x$. By transitivity of A, we have $y \in A$, and then $z \in A$. Since, by regularity, $z \neq x$, we have $x \in z$ or $z \in x$ by $\in$-connectedness. But $x \in z$ is impossible, lest $x \in z \in y \in x$, which would contradict regularity. Therefore, the only possibility is $z \in x$. This shows that x is transitive, as desired.