MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to the Second Midterm

Problem 1 For the sake of a contradiction, suppose that $D \not\subseteq B$, i.e., the set $A = D - B$ is non-empty. By regularity, there exists $a \in A$ such that $a \cap A = \emptyset$. Then $a \in D$, and for all $x \in a$, we have $x \in D$ (because D is transitive), but $x \notin A$ (because $a \cap A = \emptyset$). It follows that $x \in B$ for all $x \in a$, and thus $a \subseteq B$. By hypothesis, we have $a \in B$, contradicting $a \in A$.

Problem 2

(a) Suppose q is a limit of s, and take an arbitrary $\epsilon > 0$. Since q is a limit of s, there exists $k \in \omega$ such that for all $n > k$, $|s_n - q| < \epsilon/2$. But then, for all $m, n > k$,

$$|s_m - s_n| = |s_m - q + q - s_n| \leqslant |s_m - q| + |q - s_n| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that s is a Cauchy sequence, as desired.

(b) There are many such examples. For instance, consider the decimal expansion of the irrational number $\sqrt{2} = 1.41421356\ldots$. Let

$$s_0 = 1, \qquad s_1 = 1.4, \qquad s_2 = 1.41, \qquad s_3 = 1.414, \qquad s_4 = 1.4142,$$

and so forth, i.e., let s_n be the decimal expansion of $\sqrt{2}$ cut off after the nth digit. Then s is a Cauchy sequence of rational numbers, but it does not converge to a rational number.

If you are looking for a more precise way of stating the definition of this sequence, you may state it as follows: let $s_n = m/10^n$, where m is the largest integer that is smaller than $\sqrt{2} \cdot 10^n$. Or: let $s_n = m/10^n$, where m is the largest integer such that $m^2 < 2 \cdot 10^{2n}$.

Problem 3 Yes, such an F must exist, although it is not necessarily unique. Notice that W can have elements that are neither 0 nor a successor; at such elements, F may be defined arbitrarily. More precisely:

(a) Let $\gamma(f, y)$ be the formula

"If f is a function with domain $\operatorname{seg} S(t)$, for some $t \in W$, then $y = g(f(t))$. Otherwise, $y = a$."

Then for each f, there exists a unique y such that $\gamma(f, y)$. In particular, γ satisfies the hypothesis of the Transfinite Recursion Theorem Schema. Thus, there exists a unique function F with domain W such that

$$\gamma(F|_{\operatorname{seg} t}, F(t))$$

for all $t \in W$. We have $\gamma(F|_{\operatorname{seg} 0}, a)$, which shows that $F(0) = a$. Also, for any $t \in W$, we have $\gamma(F|_{\operatorname{seg} S(t)}, g(F(t)))$, and thus $F(S(t)) = g(F(t))$ as desired.

(b) In general, F is not unique. The simplest counterexample is given by $W = \omega \cdot 2$, $A = \{0, 1\}$, $a = 0$, and $g(x) = x$. Let $F_1, F_2 : W \to A$ be given by

$$\begin{aligned} F_1(x) &= 0, \qquad \text{for all } x \in W, \\ F_2(x) &= \begin{cases} 0, & \text{if } x < \omega, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Then both F_1 and F_2 satisfy the recursion conditions.

Problem 4 Let $\alpha \neq 0$ be an ordinal. We distinguish two cases:

Case 1: α has a largest element x. Then for all $y \in \alpha$, $y \in x$, thus $y \in x \cup \{x\} = x^+$. This implies $\alpha \subseteq x^+$. Conversely, notice that $x \in \alpha$ and, because α is transitive, $x \subseteq \alpha$, hence $x^+ = x \cup \{x\} \subseteq \alpha$. Thus we have $\alpha = x^+$ is a successor ordinal.

Case 2: α does not have a largest element. We know that $\bigcup \alpha \subseteq \alpha$ because α is a transitive set. Conversely, take any $x \in \alpha$. Since α has no largest element, there exists y with $x \in y \in \alpha$, and thus $x \in \bigcup \alpha$. This shows that $\alpha \subseteq \bigcup \alpha$, and thus $\alpha = \bigcup \alpha$ is a limit ordinal.

To show that α cannot be a successor ordinal and a limit ordinal at the same time, observe that if $\alpha = \beta^+$, then $\bigcup \alpha = \bigcup \beta^+ = \beta \neq \alpha$ by Theorem 4E.