MATH/CSCI 2112, DISCRETE STRUCTURES I, FALL 2005
Handout 3: Natural Deduction for Quantifiers

Rules.

Forall-introduction ($\forall$I)

$$\begin{array}{l|ll}
\vdots & \vdots & \\
m & u \mid \vdots & \\
\vdots & \mid \vdots & \\
n & \mid A(u) & \\
n+1 & \forall x\, A(x) & \forall\text{I}, m\text{–}n
\end{array}$$

Forall-elimination ($\forall$E)

$$\begin{array}{l|ll}
\vdots & \vdots & \\
m & \forall x\, A(x) & \\
\vdots & \vdots & \\
n & A(t) & \forall\text{E}, m
\end{array}$$

Exists-Introduction ($\exists$I)

$$\begin{array}{l|ll}
\vdots & \vdots & \\
m & A(t) & \\
\vdots & \vdots & \\
n & \exists x\, A(x) & \exists\text{I}, m
\end{array}$$

Exists-Elimination ($\exists$E)

$$\begin{array}{l|ll}
\vdots & \vdots & \\
p & \exists x\, A(x) & \\
m & u \mid \underline{A(u)} & \\
\vdots & \mid \vdots & \\
n & \mid \varphi & \\
n+1 & \varphi & \exists\text{E}, p, m\text{–}n
\end{array}$$

In the above rules:

- in $\forall$E and $\exists$I, t is any term.
- in $\forall$I and $\exists$E, u is a *fresh variable*. Here "fresh" means that this variable does not occur anywhere else in the derivation. It may only occur in the subderivation from lines m–n. The "u" that is written between the vertical lines on line m is called a *guard* — it serves as a reminder that u must be fresh in this subderivation. *In particular, this means that no formula containing u can be imported (repeated) into lines m–n from outside lines m–n. Also, this means that u cannot occur in the formula φ in lines n and $n+1$ of $\exists$E.*
- in all rules, $A(u)$ means $S^x_u A$ (substitution of variable u for free variable x), and $A(t)$ means $S^x_t A$ (substitution of term t for free variable x). In all substitutions for a free variable, *you must change the name of any bound variables, if necessary, to avoid capture of variables within a quantifier's scope, if that could occur*. It often helps to standardize the variables apart before doing a substitution.

Examples.

$$\begin{array}{r|ll}
1 & \underline{\forall x(A(x) \rightarrow B(x))} & \\
2 & \mid \underline{\exists y\, A(y)} & \\
3 & \mid u \mid \underline{A(u)} & \\
4 & \mid \mid \forall x(A(x) \rightarrow B(x)) & \text{R}, 1 \\
5 & \mid \mid A(u) \rightarrow B(u) & \forall\text{E}, 4 \\
6 & \mid \mid B(u) & \rightarrow\text{E}, 3, 5 \\
7 & \mid \mid \exists y\, B(y) & \exists\text{I}, 6 \\
8 & \mid \exists y\, B(y) & \exists\text{E}, 2, 3\text{–}7 \\
9 & \exists y\, A(y) \rightarrow \exists y\, B(y) & \rightarrow\text{I}, 2\text{–}8
\end{array}$$

$$\begin{array}{r|ll}
1 & \mid \underline{\forall x\, P(x,x)} & \\
2 & \mid u \mid \forall x\, P(x,x) & \text{R}, 1 \\
3 & \mid \mid P(u,u) & \forall\text{E}, 2 \\
4 & \mid \mid \exists z\, P(u,z) & \exists\text{I}, 3 \\
5 & \mid \forall y\, \exists z\, P(y,z) & \forall\text{I}, 2\text{–}4 \\
6 & \forall x\, P(x,x) \rightarrow \forall y\, \exists z\, P(y,z) & \rightarrow\text{I}, 1\text{–}5
\end{array}$$

Non-examples.
Non-example 1

1	$\forall x(A(x) \rightarrow B(x))$	
2	$\quad \exists y\, A(y)$	
3	$\quad u \quad A(u)$	
4	$\quad\quad \forall x(A(x) \rightarrow B(x))$	R, 1
5	$\quad\quad A(u) \rightarrow B(u)$	$\forall$E, 4
6	$\quad\quad B(u)$	$\rightarrow$E, 3, 5
7	$\quad B(u)$	$\exists$E, 2, 3–6
8	$\exists y\, A(y) \rightarrow B(u)$	$\rightarrow$I, 2–7

WRONG, because u is not fresh in lines 3–6 (u must not occur in lines 6,7,8).

Non-example 2

1	$\quad \forall x\, P(x,x)$	
2	$\quad P(u,u)$	$\forall$E, 1
3	$\quad u \quad P(u,u)$	R, 2
4	$\quad\quad \exists z\, P(u,z)$	$\exists$I, 3
5	$\quad \forall y\, \exists z\, P(y,z)$	$\forall$I, 2–4
6	$\forall x\, P(x,x) \rightarrow \forall y\, \exists z\, P(y,z)$	$\rightarrow$I, 1–5

WRONG, because u is not fresh in lines 3–4 (u cannot be repeated past the guard from line 2 to line 3).

Non-example 3

1	$\forall x(A(x) \rightarrow \exists y\, B(x,y))$	
2	$A(y)$	
3	$A(y) \rightarrow \exists y\, B(y,y)$	$\forall$E, 1
4	$\exists y\, B(y,y)$	$\rightarrow$E, 2, 3

WRONG, because the substitution in line 3 impropertly captured the variable y in the scope of a quantifier.

1	$\forall x(A(x) \rightarrow \exists y\, B(x,y))$	
2	$A(y)$	
3	$\forall x(A(x) \rightarrow \exists z\, B(x,z))$	rename bound variables, 1
4	$A(y) \rightarrow \exists z\, B(y,z)$	$\forall$E, 1
5	$\exists z\, B(y,z)$	$\rightarrow$E, 2, 4

CORRECT, because now the variable y does not get captured in the substitution in line 4.

Problems.

Problem 1 Prove the following in natural deduction:

(a) $Q \to \forall x\, P(x) \equiv \forall x\,(Q \to P(x))$ — assume that x does not occur in Q.

(b) $\sim \exists x\, P(x) \equiv \forall y \sim P(y)$.

(c) $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\,(P(x) \wedge Q(x))$.

(d) $\forall x\, P(x) \vee \forall x\, Q(x) \vdash \forall x\,(P(x) \vee Q(x))$.

(e) $\exists x\, \forall y\, P(x, y) \vdash \forall y\, \exists x\, P(x, y)$.

(f) $\exists x\, \forall y\, P(x, y) \vdash \exists z\, P(z, z)$.

(g) $\exists x\, P(x) \vee \exists x\, Q(x) \equiv \exists x(P(x) \vee Q(x))$.

(h) $\exists x(P(x) \wedge Q(x)) \vdash \exists x\, P(x) \wedge \exists x\, Q(x)$.

(i) $\exists x\, P(x, x) \vdash \exists y\, \exists z\, P(y, z)$.

(j) $\forall x\,(A(x) \to B(x)) \vdash \exists x \sim B(x) \to \exists x \sim A(x)$.

(k) $\sim \exists x\,(A(x) \wedge B(x)) \equiv \forall x\,(A(x) \to \sim B(x))$.

(l) $\exists x\, \forall y\, P(x, y, x) \vdash \exists x\, \forall y\, \exists z\, P(x, y, z)$.

(m) $\vdash \forall x(P(x) \to \exists y\, P(y))$.

(n) $\vdash \forall x(\forall y\, P(y) \to P(x))$.

(o) $\forall x\, P(x) \vdash \exists x\, P(x)$.

(p) $\forall x(A(x) \to B(x)), \forall y(B(y) \to C(y)) \vdash \forall z(A(z) \to C(z))$.

(q) $\exists x\, A(x), \forall x\,(A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(r) $\forall x\, A(x), \exists x\,(A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(s) $\sim \exists x(A(x) \vee B(x)) \equiv \forall x \sim A(x) \wedge \forall x \sim B(x)$.

(t) $\exists x\, P(x) \to \forall y\, Q(y) \equiv \forall x\, \forall y(P(x) \to Q(y))$.

Problem 2 Prove the following by natural deduction. Note: each of these problems requires the $\sim\sim$-elimination rule.

(u) $Q \to \exists x\, P(x) \equiv \exists x\,(Q \to P(x))$ — assume that x does not occur in Q.

(v) $\sim \forall x\, P(x) \equiv \exists y \sim P(y)$.

(w) $\exists x\,(A(x) \wedge B(x)) \equiv \sim \forall x\,(A(x) \to \sim B(x))$.

(x) $\vdash \exists x(\exists y\, P(y) \to P(x))$.

(y) $\sim \forall x(A(x) \wedge B(x)) \equiv \exists x \sim A(x) \vee \exists x \sim B(x)$.

(z) $\forall x\, P(x) \to \exists y\, Q(y) \equiv \exists x\, \exists y(P(x) \to Q(y))$.

Problem 3 Prove the equivalences in Problem 1 (a), (b), (c), (g), (k), (s), (t) and Problem 2 (u), (v), (w), (y), (z) by the laws of statement algebra.

Problem 4 In Problem 1 (d), (e), (f), (h), (i), (j), (l), (o), (p), (q), (r), prove that the converse direction does not hold by giving a counterexample.