Math 4680, Topics in Logic and Computation, Winter 2012

Answers to Homework 4

For problems 1(a)-(f) and 2(v): prove the left-to-right implication, using your choice of Prawitz style or Fitch style natural deduction.

Problem 1(a) Fitch style:

1	$Q \to \forall x\, P(x)$	
2	$u \quad Q \to \forall x\, P(x)$	R, 1
3	$\quad\quad Q$	
4	$\quad\quad Q \to \forall x\, P(x)$	R, 2
5	$\quad\quad \forall x\, P(x)$	$\to$E, 3, 4
6	$\quad\quad P(u)$	$\forall$E, 5
7	$\quad Q \to P(u)$	$\to$I, 3–6
8	$\forall x(Q \to P(x))$	$\forall$I, 2–7

Prawitz style:

$$\cfrac{\cfrac{\cfrac{\cfrac{Q \to \forall x\, P(x) \qquad [Q]_1}{\forall x\, P(x)}\,(\to E)}{P(u)}\,(\forall E)}{Q \to P(u)}\,(\to I)_1}{\forall x(Q \to P(x))}\,(\forall I)$$

Problem 1(b) Fitch style:

1	$\neg\exists x\, P(x)$	
2	$u \quad \neg\exists x\, P(x)$	R, 1
3	$\quad\quad P(u)$	
4	$\quad\quad \exists x\, P(x)$	$\exists$I, 3
5	$\quad\quad \neg\exists x\, P(x)$	R, 2
6	$\quad\quad \bot$	$\neg$E, 4, 5
7	$\quad \neg P(u)$	$\neg$I, 3–6
8	$\forall y\, \neg P(y)$	$\forall$I, 3–7

Prawitz style:

$$\cfrac{\cfrac{\cfrac{\neg\exists x\, P(x) \qquad \cfrac{[P(u)]_1}{\exists x\, P(x)}\,(\exists I)}{\bot}\,(\neg E)}{\neg P(u)}\,(\neg I)_1}{\forall y\, \neg P(y)}\,(\forall I)$$

Problem 1(c) Fitch style:

1		$\forall x\, P(x) \land \forall x\, Q(x)$	
2	u	$\forall x\, P(x) \land \forall x\, Q(x)$	R, 1
3		$\forall x\, P(x)$	$\land$E, 2
4		$\forall x\, Q(x)$	$\land$E, 2
5		$P(u)$	$\forall$E, 3
6		$Q(u)$	$\forall$E, 4
7		$P(u) \land Q(u)$	$\land$I, 5, 6
8		$\forall x(P(x) \land Q(x))$	$\forall$I, 2–7

Prawitz style:

$$\cfrac{\cfrac{\cfrac{\cfrac{\forall x\, P(x) \land \forall x\, Q(x)}{\forall x.P(x)}\;(\land E)}{P(u)}\;(\forall E) \qquad \cfrac{\cfrac{\forall x\, P(x) \land \forall x\, Q(x)}{\forall x.Q(x)}\;(\land E)}{Q(u)}\;(\forall E)}{P(u) \land Q(u)}\;(\land I)}{\forall x(P(x) \land Q(x))}\;(\forall I)$$

Problem 1(d) Fitch style:

1		$\forall x\, P(x) \lor \forall x\, Q(x)$	
2		$\forall x\, P(x)$	
3	u	$\forall x\, P(x)$	R, 2
4		$P(u)$	$\forall$E, 3
5		$P(u) \lor Q(u)$	$\lor$I, 4
6		$\forall x(P(x) \lor Q(x))$	$\forall$I, 3–5
7		$\forall x\, Q(x)$	
8	v	$\forall x\, Q(x)$	R, 7
9		$Q(v)$	$\forall$E, 8
10		$P(v) \lor Q(v)$	$\lor$I, 9
11		$\forall x(P(x) \lor Q(x))$	$\forall$I, 8–10
12		$\forall x(P(x) \lor Q(x))$	$\lor$E, 1, 2–6, 7–11

Prawitz style:

$$\cfrac{\forall x\, P(x) \lor \forall x\, Q(x) \qquad \cfrac{\cfrac{\cfrac{[\forall x\, P(x)]_1}{P(u)}\;(\forall E)}{P(u) \lor Q(u)}\;(\lor I)}{\forall x(P(x) \lor Q(x))}\;(\forall I) \qquad \cfrac{\cfrac{\cfrac{[\forall x\, Q(x)]_2}{Q(v)}\;(\forall E)}{P(v) \lor Q(v)}\;(\lor I)}{\forall x(P(x) \lor Q(x))}\;(\forall I)}{\forall x(P(x) \lor Q(x))}\;(\lor E)_{1,2}$$

Problem 1(e) Fitch style:

1	$\exists x\forall y\,P(x,y)$	
2	│ v │ $\exists x\forall y\,P(x,y)$	R, 1
3	│ │ u │ $\forall y\,P(u,y)$	
4	│ │ │ $P(u,v)$	∀E, 3
5	│ │ │ $\exists x\,P(x,v)$	∃I, 4
6	│ │ $\exists x\,P(x,v)$	∃E, 2, 3–5
7	│ $\forall y\exists x\,P(x,y)$	∀I, 2–6

Prawitz style:

$$\dfrac{\dfrac{\exists x\forall y\,P(x,y) \qquad \dfrac{\dfrac{[\forall y\,P(u,y)]_1}{P(u,v)}\,(\forall E)}{\exists x\,P(x,v)}\,(\exists I)}{\exists x\,P(x,v)}\,(\exists E)_1}{\forall y\exists x\,P(x,y)}\,(\forall I)$$

Problem 1(f) Fitch style:

1	$\exists x\,\forall y\,P(x,y)$	
2	│ u │ $\forall y\,P(u,y)$	
3	│ │ $P(u,u)$	∀E, 2
4	│ │ $\exists z\,P(z,z)$	∃I, 3
5	│ $\exists z\,P(z,z)$	∃E, 1, 2–5

Prawitz style:

$$\dfrac{\exists x\,\forall y\,P(x,y) \qquad \dfrac{\dfrac{[\forall y\,P(u,y)]_1}{P(u,u)}\,(\forall E)}{\exists z\,P(z,z)}\,(\exists I)}{\exists z\,P(z,z)}\,(\exists E)_1$$

Problem 2(v) Fitch style:

1	$\neg\,\forall x\,P(x)$	
2	│ │ $\neg\,\exists y\,\neg\,P(y)$	
3	│ │ u │ │ $\neg\,P(u)$	
4	│ │ │ │ $\exists y\,\neg\,P(y)$	∃I, 3
5	│ │ │ │ $\neg\,\exists y\,\neg\,P(y)$	R, 2
6	│ │ │ │ $\perp$	¬E, 4, 5
7	│ │ │ $P(u)$	C, 3–6
8	│ │ $\forall x\,P(x)$	∀I, 3–7
9	│ │ $\neg\,\forall x\,P(x)$	R, 1
10	│ │ $\perp$	¬E, 8, 9
11	│ $\exists y\,\neg\,P(y)$	C, 2–10

Prawitz style:

$$\dfrac{\dfrac{\neg\,\forall x\,P(x) \qquad \dfrac{\dfrac{\dfrac{[\neg\,\exists y\,\neg\,P(y)]_2 \qquad \dfrac{[\neg\,P(u)]_1}{\exists y\,\neg\,P(y)}\,(\exists I)}{\perp}\,(\neg\,E)}{P(u)}\,(\text{contra})_1}{\forall x\,P(x)}\,(\forall I)}{\perp}\,(\neg\,E)}{\exists y\,\neg\,P(y)}\,(\text{contra})_2$$

Problem 3(d) Informally: $P(x) = x$ is a cat, $Q(x) = x$ is a dog. If everyone is a cat or a dog, this does not imply that everyone is a cat or everyone is a dog.

Formally: Consider a structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{c, d\}$, and with $P(c) = T$, $P(d) = F$, $Q(c) = T$, and $Q(d) = F$. Then $\mathfrak{A} \models \forall x(P(x) \vee Q(x))$ but $\mathfrak{A} \not\models \forall x\, P(x) \vee \forall x\, Q(x)$.

Problem 3(j) Consider a structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{c, d\}$ with $A(c) = T$, $A(d) = F$, $B(c) = T$, and $B(d) = F$. Then $\exists x \neg A(x)$ is true, therefore $\mathfrak{A} \models \exists x\, \neg B(x) \rightarrow \exists x\, \neg A(x)$. On the other hand, since $A(c) = T$ and $B(c) = F$, we have $\mathfrak{A} \not\models A(c) \rightarrow B(c)$, therefore $\mathfrak{A} \not\models \forall x\, (A(x) \rightarrow B(x))$.

Problem 3(l) Consider a structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{1, 2\}$ and $P(1, 1, 1) = T$ and $P(1, 2, 2) = T$, and $P(x, y, z) = F$ in all other cases. Then $\mathfrak{A} \models \exists x\, \forall y\, \exists z\, P(x, y, z)$, because we can take $x = 1$, and then for every y, we can take $z = y$. On the other hand, $\mathfrak{A} \not\models \exists x\, \forall y\, P(x, y, x)$, because neither $x = 1$ nor $x = 2$ satisfies this formula.

Problem p.114 #1 Let $\mathfrak{A}$ be a fixed non-trivial group, and let $T = \{\sigma \mid \mathfrak{A} \models \sigma\}$ be the theory of $\mathfrak{A}$. Recall that the language of groups has a constant symbol e (unit), a binary function symbol $\cdot$ (multiplication), and a unary function symbol $(-)^{-1}$ (inverse), as well as equality. Consider the sentence $\varphi(x) = (x \neq e)$. Then clearly $\exists x.\varphi(x) \in T$, because $\mathfrak{A}$ is non-trivial (i.e., has at least two elements, one of which is not e). On the other hand, there exists no constant symbol c of the language such that $\varphi(c) \in T$: the only constant symbol of the language is e, and $\varphi(e)$ is the sentence $e \neq e$, which is false in $\mathfrak{A}$, hence not in T.

Problem p.114 #2 Let $\{T_i\}_{i \in I}$ be a family of theories linearly ordered by inclusion. This means that for all $i, j \in I$, either $T_i \subseteq T_j$ or $T_j \subseteq T_i$. Let $T = \bigcup_{i \in I} T_i$. We wish to show that T is a theory. Indeed, assume that φ is a sentence with $T \vdash \varphi$. Since each derivation uses only finitely many hypotheses, there exists some finite set of sentences $\sigma_1, \ldots, \sigma_n \in T$ such that $\sigma_1, \ldots, \sigma_n \vdash \varphi$. Let $i_1, \ldots, i_n$ such that $\sigma_1 \in T_{i_1}$, …, $\sigma_n \in T_{i_n}$.

Because $\{T_i\}_{i \in I}$ is linearly ordered, there exists some $i \in \{i_1, \ldots, i_n\}$ such that $T_{i_1}, \ldots, T_{i_n} \subseteq T_i$. Then we have $\sigma_1, \ldots, \sigma_n \in T_i$. Since T_i is a theory, and $\sigma_1, \ldots, \sigma_n \vdash \varphi$, we have $\varphi \in T_i$. It follows that $\varphi \in T$. Therefore T is a theory.

Next, we must show that T extends T_i, i.e., $T_i \subseteq T$ for all $i \in I$. But this holds by definition since $T = \bigcup_{i \in I} T_i$.

Finally, we must show that if each T_i is consistent, then so is T. But if T_i is consistent, then $\perp \notin T_i$. Since this holds for all $i \in I$, we have $\perp \notin T$ by definition of union. Hence T is consistent.