A Completeness Theorem for Injectivity Logic

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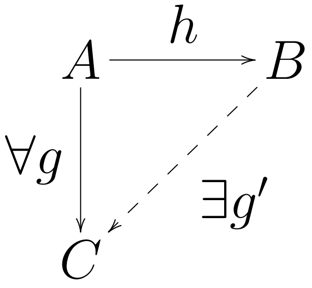
C is *h-injective* **is written** $C \models h$



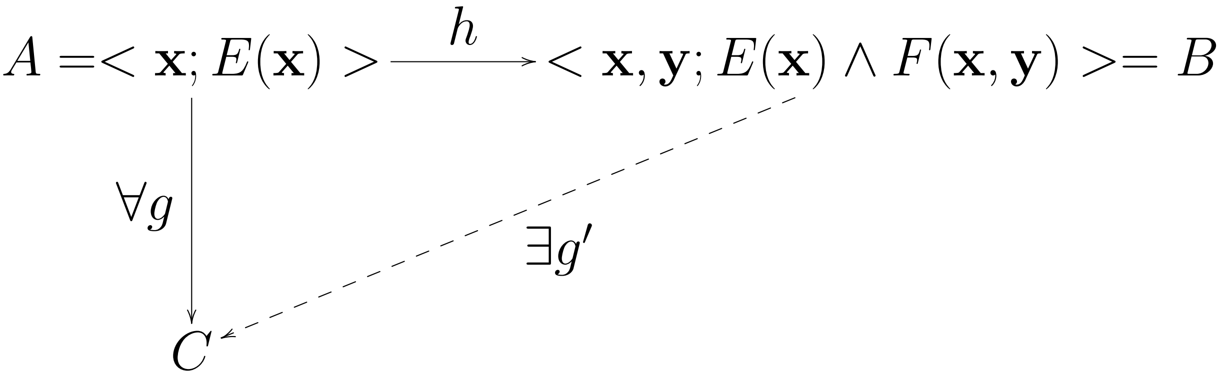
$C \in \mathcal{H}^{\triangle}$ **is written** $C \models \mathcal{H}$

$$(= \forall h \in \mathcal{H} \; (\; C \models h \;))$$

$f \in (\mathcal{H}^{\triangle})^{\triangledown}$ **is written** $\mathcal{H} \models f$

$$\forall C \; (\; C \models \mathcal{H} \;\Rightarrow\; C \models f \;)$$

<u>EXAMPLE</u>: In **Alg(Σ)** (Σ a signature), any h can be "presented by generators and relations":



(E, F sets of equations (i.e., $\in \wedge Atomic$))

$$\boxed{C \models h \quad \textbf{means} \quad C \models \forall \mathbf{x}(E(\mathbf{x}) \to \exists \mathbf{y} F(\mathbf{x}, \mathbf{y}))}$$

If A and B are finitely presentable, h "is" a (regular) finitary sentence.

Conversely, any regular sentence "is" a morphism.

$$\boxed{C \models \mathcal{H} \quad \textbf{means} \quad \forall h \in \mathcal{H} \; (\; C \models h \;)}$$

$$\boxed{\mathcal{H} \models f \quad \textbf{means} \quad \forall C \; (\; C \models \mathcal{H} \; \Rightarrow \; C \models f \;)}$$

CONTEXT:

$\mathcal{A}$ can be locally presentable, or **Top**, or...

QUESTIONS: Given $\mathcal{H} \models f$,

(1) Can we "deduce" (= construct) f from $\mathcal{H}$?

(2) If $\mathcal{H}$ and f are "finitary", is there a "finitary" proof?

ANSWERS:

(1) Yes for all sets $\mathcal{H}$ of morphisms: this follows directly from the "Small-Object Argument" ([Quillen, 67], [Ad-Her-Ros-Tho, 02]) (see below)

(2) Yes (our main result). This will give in particular a Compactness Theorem:

$$\mathcal{H} \models f \;\Rightarrow\; \mathcal{H}' \models f$$
$$\text{for some finite } \mathcal{H}' \subset \mathcal{H}$$

(will extend to a λ-ary version)

(1) **Proof.**

Note first:

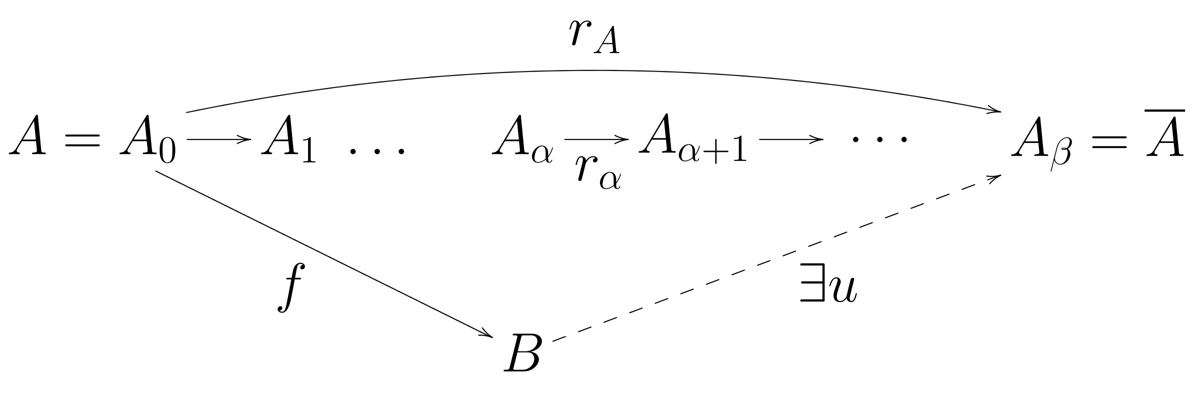
(a) Mod($\mathcal{H}$) ($= \mathcal{H}^{\triangle}$) is weakly reflective in $\mathcal{A}$.

(b) the reflectors $r_A \colon A \to \overline{A}$ are cellularly generated by $\mathcal{H}$:

$$r_A \in cell(\mathcal{H}) = \text{Comp}(\text{P.O.}(\mathcal{H}))$$

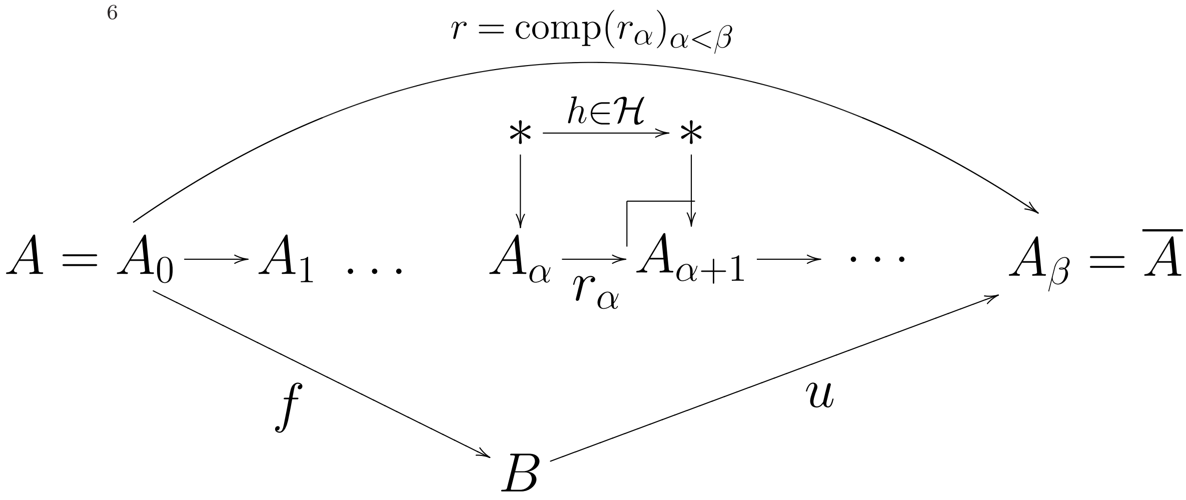
i.e., r_A is the colimit of a smooth chain of pushouts of members of $\mathcal{H}$ (i.e., all $r_\alpha \colon A_\alpha \to A_{\alpha+1}$ below are in P.O.($\mathcal{H}$))

Hence, given $\mathcal{H} \models f \colon A \to B$, we have



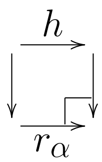
(since $\overline{A} \models \mathcal{H} \models f$).

Hence f is “deduced” from $\mathcal{H}$ using the rules:



Injectivity Deduction System $(\vdash_\infty)$

TRANSFINITE COMPOSITION $\dfrac{r_\alpha \ (\alpha < \beta)}{r}$ if $r = \mathrm{comp}(r_\alpha)_{\alpha<\beta}$, β is any ordinal

PUSHOUT $\dfrac{h}{r_\alpha}$ if 

CANCELLATION $\dfrac{u \cdot f}{f}$ if $u \cdot f$ is defined

We write this as

$$\mathcal{H} \vdash_\infty f$$

Soundness ($\mathcal{H} \vdash_\infty f \Rightarrow \mathcal{H} \models f$) is straightforward, hence:

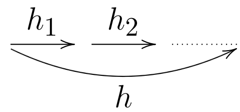
$$\mathcal{H} \models f \text{ iff } \mathcal{H} \vdash_\infty f$$

for every set $\mathcal{H}$ and every f

[λ-ary] Injectivity Deduction System ~~($\vdash_\infty$)~~ [$\vdash_\lambda$]

[λ-ARY] ~~TRANSFINITE~~ COMPOSITION

$$\frac{h_\alpha \ (\alpha < \beta)}{h}$$

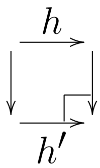


~~β is any ordinal~~
[$\beta < \lambda$]

PUSHOUT

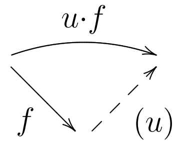
$$\frac{h}{h'}$$

if



CANCELLATION

$$\frac{u \cdot f}{f}$$



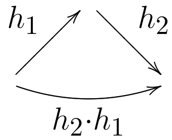
(2) **Definitions**:

Finitary proof ($\mathcal{H} \vdash_\omega f$): if f can be obtained from $\mathcal{H}$ by a finite number of applications of the rules:

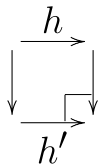
Finitary Injectivity Deduction System $(\vdash_\omega)$

IDENTITY $\dfrac{}{\mathrm{id}_A}$

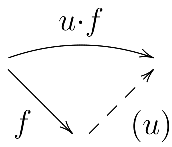
COMPOSITION $\dfrac{h_1 \; h_2}{h_2 \cdot h_1}$



PUSHOUT $\dfrac{h}{h'}$



CANCELLATION $\dfrac{u \cdot f}{f}$



$f \colon A \to B$ is finitary if A and B are finitely presentable ($\neq$ "f is finitely presentable").

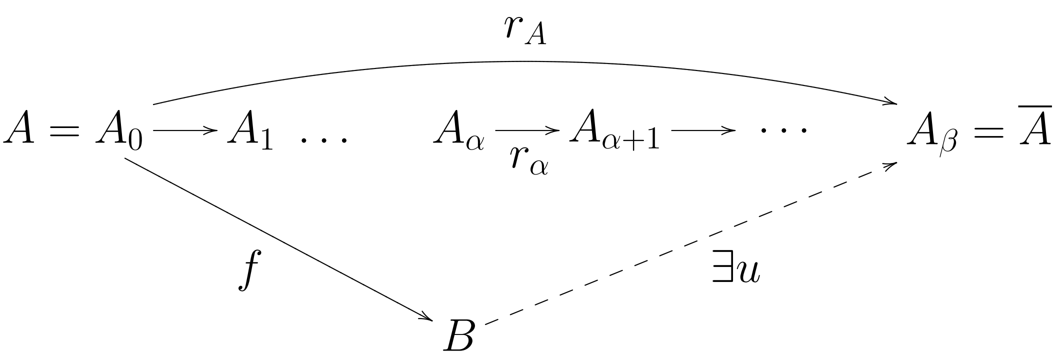
Theorem

(When f and all $h \in \mathcal{H}$ finitary)

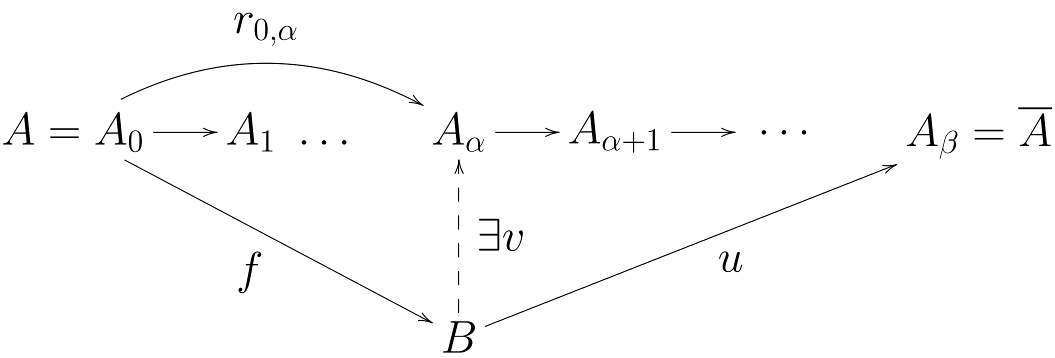
$$\mathcal{H} \models f \quad \text{iff} \quad \mathcal{H} \vdash_{\omega} f$$

Proof. (Assume $\mathcal{A}$ locally finitely presentable)

As before, $\overline{A} \models \mathcal{H} \models f \colon A \to B$ gives:



This time A and B are finitely presentable, so:



for some α

However $\mathcal{H} \not\vdash_{\omega} r_{0,\alpha}$!

The wanted deduction is not (quite) part of this diagram.

We know that the class of ordinals

$S = \{\alpha \mid \text{some}\,\alpha\text{-chain in P.O.}(\mathcal{H}) \text{ factorizes through } f\}$

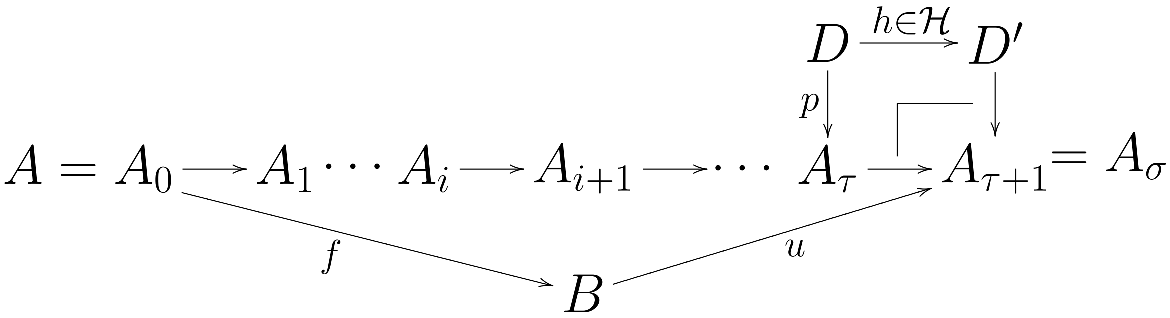
is not empty, hence it has a first element σ.

We show that σ is finite:

Suppose σ is infinite.

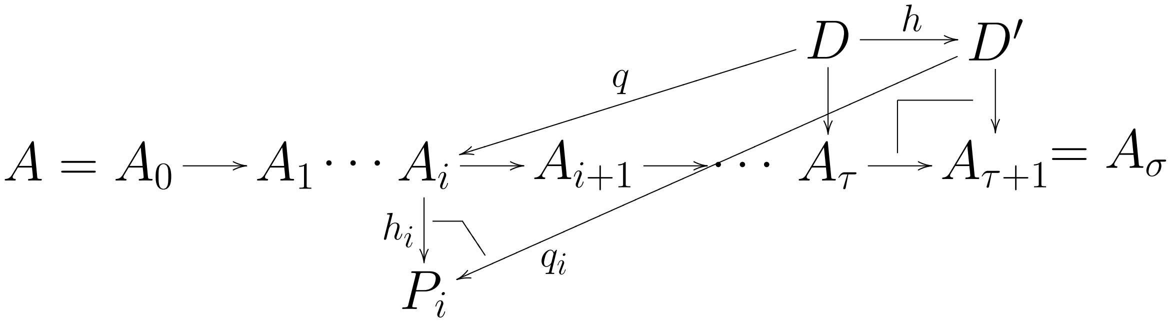
Then $\sigma = \tau + k$ for τ limit ordinal and k finite.

- $k \neq 0$ (because A, B are finitely presentable)
- We can assume $k = 1$.

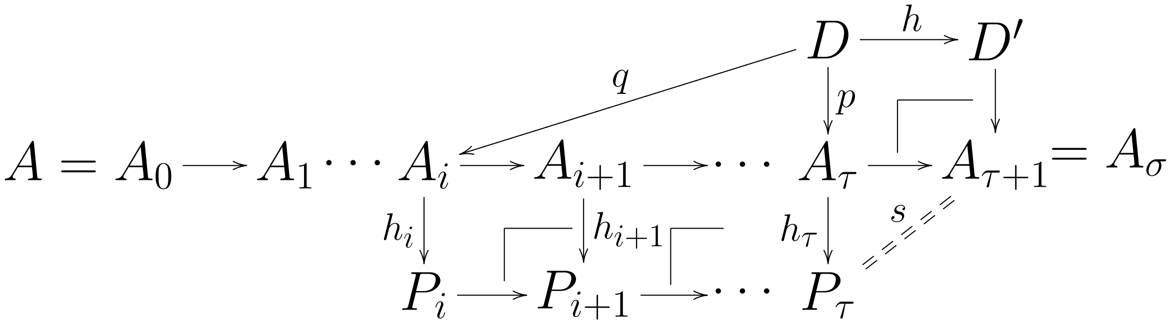


Then p factorizes through the chain by some q (because D is finitely presentable)

Let $(h_i, q_i) = \text{Pushout}(h, q)$:



Then take successive pushouts, and their colimits, etc.:



Then there exists an isomorphism s making the triangle commute, since $h_\tau (= \mathrm{colim}(h_j)_{j \geq i})$ is also the pushout of h by p!

But then the smooth τ-chain in P.O.$(\mathcal{H})$

$$A \to A_1 \to \cdots A_i \xrightarrow{h_i} P_i \to P_{i+1} \to \cdots P_\tau$$

factorizes through f,
contradicting the minimality of σ.

EXAMPLES AND COUNTEREXAMPLES

1) The Finitary Completeness Theorem

$$\mathcal{H} \models f \quad \Leftrightarrow \quad \mathcal{H} \vdash_\omega f$$

holds in all *weakly locally ranked* categories (the proof is more involved).

2) In locally finitely presentable categories,

$$\mathcal{H} \models_\omega f \quad \not\Rightarrow \quad \mathcal{H} \vdash_\omega f.$$

in general (Here $\mathcal{H} \models_\omega f$ means $\mathcal{H} \models f$ in $\mathcal{A}_{fp}$)

3) In **CPO(1)** (= continuous posets with an extra binary relation),

$$\mathcal{H} \models f \quad \not\Rightarrow \quad \mathcal{H} \vdash_\infty f$$

($\mathcal{H}$ a set) in general.

4) In locally finitely presentable categories, the (∞-ary) Completeness Theorem

$$\mathcal{H} \models f \quad \Leftrightarrow \quad \mathcal{H} \vdash_\infty f$$

does NOT hold for CLASSES $\mathcal{H}$ in general.

However it holds for classes $\mathcal{H}$ made of
(a) epimorphisms (easy), or of
(b) finitely presentable morphisms (less easy).