Co-rings Over Operads

Kathryn Hess

Institute of Geometry, Algebra and Topology
Ecole Polytechnique Fédérale de Lausanne

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Slogan

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- parametrize n-ary operations, and
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- parametrize higher, “up to homotopy” structure on homomorphisms, and
- govern the relations among the “higher homotopies" and the n-ary operations.

Co-rings over operads should therefore be considered as relative operads.

Notation and conventions

- **Ch** is the category of chain complexes over a commutative ring R that are bounded below.
- **Ch** is closed, symmetric monoidal with respect to the tensor product:

$$(C, d) \otimes (C', d') := (C'', d'')$$

where

$$C''_n = \bigoplus_{i+j=n} C_i \otimes_R C'_j$$

and

$$d'' = d \otimes_R C' + C \otimes_R d'.$$

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- (Co)monoids in a given monoidal category are not assumed to be (co)unital.

Outline

The cobar construction

Let **C** denote the category of chain coalgebras, i.e., of comonoids in **Ch**. Let **A** denote the category of chain algebras, i.e., of monoids in **Ch**.

The cobar construction is a functor

$$\Omega : \mathbf{C} \longrightarrow \mathbf{A} : C \longmapsto \Omega C = \big(T(s^{-1}C), d_\Omega\big),$$

where

- T is the free monoid functor on graded R-modules,
- $(s^{-1}C)_n = C_{n+1}$ for all n, and
- d_Ω is the derivation specified by

$$d_\Omega s^{-1} = -s^{-1}d + (s^{-1} \otimes s^{-1})\Delta,$$

where d and Δ are the differential and coproduct on C.

The category **DCSH**

- $\mathrm{Ob}\,\mathbf{DCSH} = \mathrm{Ob}\,\mathbf{C}$.
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The chain map $\varphi_1 : C \to C'$ is called a DCSH-map.

Topological significance I

Let K be a simplicial set.

Theorem (Gugenheim-Munkholm)

The natural coproduct $\Delta_K : C_*K \to C_*K \otimes C_*K$ *is naturally a DCSH-map*

Thus, there exists $\varphi_K \in \mathbf{A}\Big(\Omega C_*K, \Omega\big(C_*K \otimes C_*K\big)\Big)$ such that $(\varphi_K)_1 = \Delta_K$.

Topological significance II

Theorem (H.-Parent-Scott-Tonks)

There is a natural, coassociative coproduct ψ_K on $\Omega C_* K$, given by the composite

$$\Omega C_* K \xrightarrow{\varphi_K} \Omega\big(C_* K \otimes C_* K\big) \xrightarrow{q} \Omega C_* K \otimes \Omega C_* K,$$

where q is Milgram's natural transformation.

Furthermore, Szczarba's natural equivalence of chain algebras

$$Sz : \Omega C_* K \xrightarrow{\simeq} C_* GK$$

is a DCSH-map with respect to ψ_K and to the natural coproduct Δ_{GK} on $C_* GK$.

Monoidal products of bimodules

Let $(\mathbf{M}, \otimes, I)$ be a bicomplete monoidal category. Let (A, μ) be a monoid in $\mathbf{M}$.

Remark

The category of A-bimodules is also monoidal, with monoidal product $\underset{A}{\otimes}$ given by the coequalizer

$$M \otimes A \otimes N \underset{M\otimes\lambda}{\overset{\rho\otimes N}{\rightrightarrows}} M \otimes N \longrightarrow M \underset{A}{\otimes} N.$$

Definition of co-rings

An A-co-ring is a comonoid (R, ψ) in the category of A-bimodules, i.e.,

$$\psi : R \longrightarrow R \underset{A}{\otimes} R$$

is coassociative.

$\mathbf{CoRing}_A$ is the category of A-co-rings and their morphisms.

Example: the canonical co-ring

Let $\varphi : B \to A$ be a monoid morphism.

Let $R = A \underset{B}{\otimes} A$.

Define $\psi : R \to R \underset{A}{\otimes} R$ to be following composite of A-bimodule maps.

$$\begin{array}{ccc} A \underset{B}{\otimes} A & \xrightarrow{\cong} & A \underset{B}{\otimes} B \underset{B}{\otimes} A \\ \Big\downarrow \psi & & \Big\downarrow A \underset{B}{\otimes} \varphi \underset{B}{\otimes} A \\ (A \underset{B}{\otimes} A) \underset{A}{\otimes} (A \underset{B}{\otimes} A) & \xleftarrow{\cong} & A \underset{B}{\otimes} A \underset{B}{\otimes} A \end{array}$$

This example arose in Galois theory.

Kleisli constructions I

- ${}_{(A,R)}\mathbf{Mod}$
  - $\mathrm{Ob}\,{}_{(A,R)}\mathbf{Mod} = \mathrm{Ob}\,{}_{A}\mathbf{Mod}$
  - ${}_{(A,R)}\mathbf{Mod}(M,N) = {}_{A}\mathbf{Mod}(R \underset{A}{\otimes} M, N)$
  - Composition of $\varphi \in {}_{(A,R)}\mathbf{Mod}(M,M')$ and $\varphi' \in {}_{(A,R)}\mathbf{Mod}(M',M'')$ given by the composite of left A-module morphisms below.

$$R \underset{A}{\otimes} M \xrightarrow{\psi \underset{A}{\otimes} M} R \underset{A}{\otimes} R \underset{A}{\otimes} M \xrightarrow{R \underset{A}{\otimes} \varphi} R \underset{A}{\otimes} M' \xrightarrow{\varphi'} M''$$

Kleisli constructions II

- $\mathbf{Mod}_{(A,R)}$
  - Ob $\mathbf{Mod}_{(A,R)} =$ Ob $\mathbf{Mod}_A$
  - $\mathbf{Mod}_{(A,R)}(M,N) = \mathbf{Mod}_A(M \underset{A}{\otimes} R, N)$
  - Composition of $\varphi \in \mathbf{Mod}_{(A,R)}(M,M')$ and $\varphi' \in \mathbf{Mod}_{(A,R)}(M',M'')$ given by the composite of right A-module morphisms below.

$$M \underset{A}{\otimes} R \xrightarrow{M \underset{A}{\otimes} \psi} M \underset{A}{\otimes} R \underset{A}{\otimes} R \xrightarrow{\varphi \underset{A}{\otimes} R} M' \underset{A}{\otimes} R \xrightarrow{\varphi'} M''$$

Symmetric sequences

Let $(\mathbf{M}, \otimes, I)$ be a bicomplete, closed, symmetric monoidal category.

$\mathbf{M}^{\Sigma}$ is the category of symmetric sequences in $\mathbf{M}$.

$$\mathcal{X} \in \mathrm{Ob}\,\mathbf{M}^{\Sigma} \Longrightarrow \mathcal{X} = \{\mathcal{X}(n) \in \mathrm{Ob}\,\mathbf{M} \mid n \geq 0\},$$

where $\mathcal{X}(n)$ admits a right action of the symmetric group Σ_n, for all n.

The level monoidal structure

Let

$$- \otimes - : \mathbf{M}^{\Sigma} \times \mathbf{M}^{\Sigma} \longrightarrow \mathbf{M}^{\Sigma}$$

be the functor given by $(\mathcal{X} \otimes \mathcal{Y})(n) = \mathcal{X}(n) \otimes \mathcal{Y}(n)$, with diagonal Σ_n-action.

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Proposition

$(\mathbf{M}^{\Sigma}, \otimes, \mathcal{C})$ *is a closed, symmetric monoidal category, where* $\mathcal{C}(n) = I$ *with trivial* Σ_n*-action, for all* n*.*

The graded monoidal structure

Let

$$- \odot - : \mathbf{M}^{\Sigma} \times \mathbf{M}^{\Sigma} \longrightarrow \mathbf{M}^{\Sigma}$$

be the functor given by

$$(\mathcal{X} \odot \mathcal{Y})(n) = \coprod_{i+j=n} \big(\mathcal{X}(i) \otimes \mathcal{Y}(j)\big) \underset{\Sigma_i \times \Sigma_j}{\otimes} I[\Sigma_n],$$

where $I[\Sigma_n]$ is the free Σ_n-object on I.

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where $I[\Sigma_n]$ is the free Σ_n-object on I.

Proposition

$(\mathbf{M}^{\Sigma}, \odot, \mathcal{U})$ *is a closed, symmetric monoidal category, where* $\mathcal{U}(0) = I$ *and* $\mathcal{U}(n) = O$ *(the 0-object), for all* $n > 0$.

The composition monoidal structure

Let

$$- \circ - : \mathbf{M}^{\Sigma} \times \mathbf{M}^{\Sigma} \longrightarrow \mathbf{M}^{\Sigma}$$

be the functor given by

$$(\mathcal{X} \circ \mathcal{Y})(n) = \coprod_{m \geq 0} \mathcal{X}(m) \underset{\Sigma_m}{\otimes} (\mathcal{Y}^{\odot m})(n).$$

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Proposition

$(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$ is a right-closed, monoidal category, where $\mathcal{J}(1) = I$ and $\mathcal{J}(n) = O$ (the 0-object), for all $n \neq 1$.

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Proposition

There is a natural transformation

$$\iota : (\mathcal{X} \otimes \mathcal{Y}) \circ (\mathcal{X}' \otimes \mathcal{Y}') \longrightarrow (\mathcal{X} \circ \mathcal{X}') \otimes (\mathcal{Y} \circ \mathcal{Y}'),$$

called the intertwiner.

Operads

An operad in **M** is a unital monoid $(\mathcal{P}, \gamma, \eta)$ in $(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$.

Operads

An operad in $\mathbf{M}$ is a unital monoid $(\mathcal{P}, \gamma, \eta)$ in $(\mathbf{M}^{\Sigma}, \circ, \mathcal{J})$.

More explicitly, there is a family of morphisms in $\mathbf{M}$

$$\gamma_{\vec{n}} : \mathcal{P}(k) \otimes \Big(\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)\Big) \to \mathcal{P}\big(\sum_{i=1}^{k} n_i\big),$$

for all $k \geq 0$ and all $\vec{n} = (n_1, ..., n_k) \in \mathbb{N}^k$, that are appropriately equivariant, associative and unital.

Examples of operads I

The associative operad $\mathcal{A}$.

For all $n \in \mathbb{N}$,

$$\mathcal{A}(n) = I[\Sigma_n],$$

on which Σ_n acts by right multiplication, and

$$\gamma_{\vec{n}} : \mathcal{A}(k) \otimes \Big(\mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_k)\Big) \to \mathcal{A}\big(\sum_{i=1}^{k} n_i\big)$$

is given by "block permutation."

Examples of operads II

The endomorphism operad $\mathcal{E}_X$ and the coendomorphism operad $\widehat{\mathcal{E}}_X$ on an object X of $\mathbf{M}$.

Let $\hom(Y, -)$ denote the right adjoint to $- \otimes Y$.

For all $n \in \mathbb{N}$,

$$\mathcal{E}_X(n) = \hom(X^{\otimes n}, X) \quad \text{and} \quad \widehat{\mathcal{E}}_X(n) = \hom(X, X^{\otimes n})$$

on which Σ_n acts by permuting inputs/outputs, and

$$\gamma_{\vec{n}} : \mathcal{E}_X(k) \otimes \Big(\mathcal{E}_X(n_1) \otimes \cdots \otimes \mathcal{E}_X(n_k)\Big) \to \mathcal{E}_X\big(\sum_{i=1}^{k} n_i\big)$$

$$\gamma_{\vec{n}} : \widehat{\mathcal{E}}_X(k) \otimes \Big(\widehat{\mathcal{E}}_X(n_1) \otimes \cdots \otimes \widehat{\mathcal{E}}_X(n_k)\Big) \to \widehat{\mathcal{E}}_X\big(\sum_{i=1}^{k} n_i\big)$$

are given by "composition of functions".

Algebras and coalgebras over operads

Let $(\mathcal{P}, \gamma, \eta)$ be an operad in **M**.

- A $\mathcal{P}$-algebra consists of an object A of **M**, together with a morphism of operads $\mu : \mathcal{P} \to \mathcal{E}_A$.

- A $\mathcal{P}$-coalgebra consists of an object C of **M**, together with a morphism of operads $\delta : \mathcal{P} \to \widehat{\mathcal{E}}_C$.

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$$\iff \exists \, \{\mu_n : \mathcal{P}(n) \otimes A^{\otimes n} \to A\}_{n \geq 0}$$

–appropriately equivariant, associative and unital.

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Morphisms of $\mathcal{P}$-(co)algebras

- A $\mathcal{P}$-algebra morphism from (A, μ) to (A', μ') is a morphism $\varphi : A \to A'$ in **M** such that the following diagram commutes for all n.

$$\begin{array}{ccc} \mathcal{P}(n) \otimes A^{\otimes n} & \xrightarrow{\mu_n} & A \\ \downarrow{\scriptstyle \mathcal{P}(n)\otimes\varphi^{\otimes n}} & & \downarrow{\scriptstyle \varphi} \\ \mathcal{P}(n) \otimes (A')^{\otimes n} & \xrightarrow{\mu'_n} & A' \end{array}$$

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- A $\mathcal{P}$-coalgebra morphism from (C, δ) to (C', δ') is a morphism $\varphi : C \to C'$ in **M** such that the following diagram commutes for all n.

$$\begin{array}{ccc} C \otimes \mathcal{P}(n) & \xrightarrow{\delta_n} & C^{\otimes n} \\ \downarrow{\scriptstyle \varphi\otimes\mathcal{P}(n)} & & \downarrow{\scriptstyle \varphi^{\otimes n}} \\ C' \otimes \mathcal{P}(n) & \xrightarrow{\delta'_n} & (C')^{\otimes n} \end{array}$$

$\mathcal{P}$-algebras as left $\mathcal{P}$-modules

Let $z : \mathbf{M} \to \mathbf{M}^{\Sigma}$ be the functor defined on objects by $z(X)(0) = X$ and $z(X)(n) = O$ for all $n > 0$.

Proposition (Kapranov-Manin,?)

The functor z restricts and corestricts to a full and faithful functor

$$z : \mathcal{P}\text{-}\mathbf{Alg} \to {}_{\mathcal{P}}\mathbf{Mod}.$$

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In particular, if $\mathbf{M} = \mathbf{Ch}$, then there is a full and faithful functor

$$z : \mathbf{A} \to {}_{\mathcal{A}}\mathbf{Mod},$$

since $\mathbf{A} = \mathcal{A}\text{-}\mathbf{Alg}$ in $\mathbf{Ch}$.

$\mathcal{P}$-coalgebras as right $\mathcal{P}$-modules

Let $\mathcal{T} : \mathbf{M} \to \mathbf{M}^{\Sigma}$ be the functor defined on objects by $\mathcal{T}(X)(n) = X^{\otimes n}$, where Σ_n acts by permuting factors.

Proposition (H.-Parent-Scott)

The functor $\mathcal{T}$ restricts and corestricts to a full and faithful functor

$$\mathcal{T} : \mathcal{P}\text{-}\mathbf{Coalg} \to {}_{\mathcal{A}}\mathbf{Mod}_{\mathcal{P}}.$$

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since $\mathbf{C} = \mathcal{A}\text{-}\mathbf{Coalg}$ in $\mathbf{Ch}$.

Monoids and multiplicative morphisms

Given

- $(\mathcal{M}, \mu), (\mathcal{M}', \mu')$, monoids in $(\mathbf{M}^\Sigma, \otimes, \mathcal{C})$, and
- $(\mathcal{X}, \Delta)$, a comonoid in $(\mathbf{M}^\Sigma, \otimes, \mathcal{C})$.

$\theta : \mathcal{M} \circ \mathcal{X} \to \mathcal{M}'$ is multiplicative if

$$\begin{array}{ccccccc} \mathcal{M}^{\otimes 2} \circ \mathcal{X} & \xrightarrow{\mathcal{M}^{\otimes 2} \circ \Delta} & \mathcal{M}^{\otimes 2} \circ \mathcal{X}^{\otimes 2} & \xrightarrow{\iota} & (\mathcal{M} \circ \mathcal{X})^{\otimes 2} & \xrightarrow{\theta^{\otimes 2}} & \mathcal{M}'^{\otimes 2} \\ \downarrow{\scriptstyle \mu \circ \mathcal{X}} & & & & & & \downarrow{\scriptstyle \mu'} \\ \mathcal{M} \circ \mathcal{X} & & & \xrightarrow{\theta} & & & \mathcal{M}' \end{array}$$

commutes.

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\downarrow{\scriptstyle \mu \circ \mathcal{X}} & & & & & & \downarrow{\scriptstyle \mu'} \\
\mathcal{M} \circ \mathcal{X} & & & \xrightarrow{\theta} & & & \mathcal{M}'
\end{array}$$

commutes.

Remark

If (A, μ) is a monoid in $\mathbf{M}$, then $\mathcal{T}(A)$ is a monoid in $(\mathbf{M}^{\Sigma}, \otimes, \mathcal{C})$, where the multiplication in level n is

$$A^{\otimes n} \otimes A^{\otimes n} \xrightarrow{\cong} (A \otimes A)^{\otimes n} \xrightarrow{\mu^{\otimes n}} A^{\otimes n}.$$

The diffracting functor

Comon$_\otimes$ is the category of comonoids in $(\mathbf{Ch}^\Sigma, \otimes, \mathcal{C})$.

Theorem (H.-P.-S.)

There is a functor

$$\Phi : \mathbf{Comon}_\otimes \to \mathbf{CoRing}_\mathcal{A}$$

such that the underlying $\mathcal{A}$-bimodule of $\Phi(\mathcal{X})$ is free, for all objects $\mathcal{X}$ in **Comon**$_\otimes$.

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Corollary

There are functors from $\mathbf{Comon}_{\otimes}^{op}$ to the category of small semicategories, given on objects by:

$$\mathcal{X} \longmapsto {}_{(\mathcal{A},\Phi(\mathcal{X}))}\mathbf{Mod} \quad \textit{and} \quad \mathcal{X} \longmapsto \mathbf{Mod}_{(\mathcal{A},\Phi(\mathcal{X}))}.$$

The Milgram transformation

Theorem (H.-P.-S.)

There is a natural transformation

$$q : \Phi(- \otimes -) \to \Phi(-) \otimes \Phi(-)$$

of functors from $\mathbf{Comon}_{\otimes} \times \mathbf{Comon}_{\otimes}$ *to* $\mathbf{CoRing}_{\mathcal{A}}$.

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Corollary

Let $(\mathfrak{X}, \Delta)$ *be an object in* $\mathbf{Comon}_{\otimes}$. *If* $\Delta : \mathfrak{X} \to \mathfrak{X} \otimes \mathfrak{X}$ *is a morphism in* $\mathbf{Comon}_{\otimes}$, *then* $\Phi(\mathfrak{X})$ *admits a level coproduct*

$$\Phi(\mathfrak{X}) \xrightarrow{\Phi(\Delta)} \Phi(\mathfrak{X} \otimes \mathfrak{X}) \xrightarrow{q} \Phi(\mathfrak{X}) \otimes \Phi(\mathfrak{X}).$$

Induction

Let C and C' be chain coalgebras. Let $\mathcal{X}$ be an object in $\mathbf{Comon}_{\otimes}$.

A (transposed tensor) morphism of right $\mathcal{A}$-modules

$$\theta : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \longrightarrow \mathcal{T}(C')$$

naturally induces a multiplicative morphism of symmetric sequences

$$\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \longrightarrow \mathcal{T}(\Omega C').$$

Linearization

Let C and C' be chain coalgebras. Let $\mathcal{X}$ be an object in **Comon**$_{\otimes}$.

A multiplicative morphism of symmetric sequences

$$\theta : \mathcal{T}(\Omega C) \circ \mathcal{X} \longrightarrow \mathcal{T}(\Omega C')$$

can be naturally linearized to a (transposed tensor) morphism of right $\mathcal{A}$-modules

$$\mathrm{Lin}(\theta) : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \longrightarrow \mathcal{T}(C').$$

The Cobar Duality Theorem

Theorem (H.-P.-S.)

Let $\mathbf{D}$ *be any small category. There are mutually inverse, natural isomorphisms*

$$\mathbf{Mod}^{tt}_{\mathcal{A}}\big(\mathcal{T}(-)\underset{\mathcal{A}}{\circ}\Phi(-),\mathcal{T}(-)\big)\xrightarrow{\text{Ind}}\mathbf{Ch}^{\Sigma}_{mult}\big(\mathcal{T}(\Omega-)\circ-,\mathcal{T}(\Omega-)\big)$$

and

$$\mathbf{Ch}^{\Sigma}_{mult}\big(\mathcal{T}(\Omega-)\circ-,\mathcal{T}(\Omega-)\big)\xrightarrow{\text{Lin}}\mathbf{Mod}^{tt}_{\mathcal{A}}\big(\mathcal{T}(-)\underset{\mathcal{A}}{\circ}\Phi(-),\mathcal{T}(-)\big)$$

of functors from $\mathbf{C}^{\mathbf{D}}\times\mathbf{Comon}_{\otimes}\times\mathbf{C}^{\mathbf{D}}$ *to* $\mathbf{Set}^{\mathbf{D}}$.

Acyclic models

Let **D** be a category, and let $\mathfrak{M}$ be a set of objects in **D**. Let $X : \mathbf{D} \to \mathbf{Ch}$ be a functor.

- X is free with respect to $\mathfrak{M}$ if there is a set $\{x_\mathfrak{m} \in X(\mathfrak{m}) \mid \mathfrak{m} \in \mathfrak{M}\}$ such that

$$\{X(f)(x_\mathfrak{m}) \mid f \in \mathbf{D}(\mathfrak{m}, d), \mathfrak{m} \in \mathfrak{M}\}$$

  is a $\mathbb{Z}$-basis of $X(d)$ for all objects d in **D**.
- X is acyclic with respect to $\mathfrak{M}$ if $X(\mathfrak{m})$ is acyclic for all $\mathfrak{m} \in \mathfrak{M}$.

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- X is acyclic with respect to $\mathfrak{M}$ if $X(\mathfrak{m})$ is acyclic for all $\mathfrak{m} \in \mathfrak{M}$.

More generally, if **C** is a category with a forgetful functor U to **Ch** and $X : \mathbf{D} \to \mathbf{C}$ is a functor, we say that X is free, respectively acyclic, with respect to $\mathfrak{M}$ if UX is.

The existence theorem

Theorem (H.-P.-S.)

Let **D** *be a small category, and let* $F, G : \mathbf{D} \to \mathbf{C}$ *be functors. Let* $U : \mathbf{C} \to \mathbf{Ch}$ *be the forgetful functor.*

If there is a set of models in **D** *with respect to which* F *is free and* G *is acyclic, then for all level comonoids* $\mathcal{X}$ *under* $\mathcal{J}$ *and for all natural transformations* $\tau : UF \to UG$*, there exists a multiplicative natural transformation*

$$\theta_{\mathcal{X}} : \mathcal{T}(\Omega F) \circ \mathcal{X} \to \mathcal{T}(\Omega G)$$

extending $s^{-1}\tau$.

Proof of the existence theorem

Proof.

Since $\Phi(\mathfrak{X})$ admits a particularly nice differential filtration, acyclic models methods suffice to prove the existence of a (transposed tensor) natural transformation

$$\widehat{\tau}_{\mathfrak{X}} : \mathcal{T}(F) \underset{\mathcal{A}}{\circ} \Phi(\mathfrak{X}) \to \mathcal{T}(G),$$

extending τ.

We can then apply the Cobar Duality Theorem and set $\theta_{\mathfrak{X}} = \mathrm{Ind}(\widehat{\tau}_{\mathfrak{X}})$. □

The Alexander-Whitney co-ring

The Alexander-Whitney co-ring is $\mathfrak{F} = \Phi(\mathfrak{J})$.

The level coproduct $\mathfrak{J} \to \mathfrak{J} \otimes \mathfrak{J}$ is composed of the isomorphisms $I \xrightarrow{\cong} I \otimes I$ and $O \xrightarrow{\cong} O \otimes O$.

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Theorem (H.-P.-S.)

- *$\mathfrak{F}$ admits a counit $\varepsilon : \mathfrak{F} \to \mathcal{A}$ inducing a homology isomorphism in each level. (In fact, $\mathfrak{F}$ is exactly the two-sided Koszul resolution of $\mathcal{A}$.)*
- *$\mathfrak{F}$ admits a coassociative, level coproduct, i.e., $\mathfrak{F}$ is an object in* $\mathbf{Comon}_{\otimes}$.

An operadic description of **DCSH**

Theorem (H.-P.-S.)

DCSH *is isomorphic to* $(\mathcal{A}, \mathcal{F})$-**Coalg**, *where*

- $\mathrm{Ob}\,(\mathcal{A}, \mathcal{F})$-**Coalg** $= \mathrm{Ob}\,$**C**, *and*
- $(\mathcal{A}, \mathcal{F})$-**Coalg**$(C, C') =$ **Mod**$_{(\mathcal{A},\mathcal{F})}\big(\mathcal{T}(C), \mathcal{T}(C')\big)$.

Remark

$(\mathcal{A}, \mathcal{F})$-**Coalg** inherits a monoidal structure from the level comonoidal structure of $\mathcal{F}$.

Existence of DCSH maps

Theorem (H.-P.-S.)

Let $\mathbf{D}$ *be a small category, and let* $F, G : \mathbf{D} \to \mathbf{C}$ *be functors. Let* $U : \mathbf{C} \to \mathbf{Ch}$ *be the forgetful functor.*

If there is a set of models in $\mathbf{D}$ *with respect to which* F *is free and* G *is acyclic, then for all natural transformations* $\tau : UF \to UG$*, there exists a natural transformation of functors into* $\mathbf{A}$

$$\theta_{\mathfrak{X}} : \Omega F \to \Omega G$$

extending $s^{-1}\tau$

Thus, for all $d \in \mathrm{Ob}\, D$*, the chain map*

$$\tau_d : UF(d) \to UG(d)$$

is naturally a DCSH-map.

Consequences

- In [H.-P.-S.-T.] this theorem is applied to proving that $\psi_K : \Omega C_* K \to \Omega C_* K \otimes \Omega C_* K$ is a DCSH-map.
- This theorem has also been applied to proving the existence of crucial DCSH-structures in constructions of models of homotopy orbit spaces and of double loop spaces.

The problem

Let $\theta : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \to \mathcal{T}(C')$ be a transposed tensor morphism, where C and C' are chain coalgebras, and $\mathcal{X}$ is a level comonoid. Let $\mathcal{P}$ be an operad in **Ch**.

Questions

- If $\mathcal{X}$ is a right $\mathcal{P}$-module and $\Omega C'$ is a $\mathcal{P}$-coalgebra, when is $\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \to \mathcal{T}(\Omega C')$ a morphism of right $\mathcal{P}$-modules?
- If $\mathcal{X}$ is a left $\mathcal{P}$-module and ΩC is a $\mathcal{P}$-coalgebra, when does $\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \to \mathcal{T}(\Omega C')$ induce a morphism

$$\widehat{\mathrm{Ind}(\theta)} : \mathcal{T}(\Omega C) \underset{\mathcal{P}}{\circ} \mathcal{X} \to \mathcal{T}(\Omega C')?$$

Preliminaries on Hopf operads

- A Hopf operad is a level comonoid in the category of operads, i.e., an operad $\mathcal{P}$ endowed with a morphism of operads $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$.

Preliminaries on Hopf operads

- A Hopf operad is a level comonoid in the category of operads, i.e., an operad $\mathcal{P}$ endowed with a morphism of operads $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$.
- If $(\mathcal{P}, \Delta)$ is a Hopf operad, then ${}_{\mathcal{P}}\mathbf{Mod}$, $\mathbf{Mod}_{\mathcal{P}}$ and ${}_{\mathcal{P}}\mathbf{Mod}_{\mathcal{P}}$ are monoidal with respect to the level monoidal product $\otimes$.

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- If $(\mathcal{P}, \Delta)$ is a Hopf operad, then ${}_{\mathcal{P}}\mathbf{Mod}$, $\mathbf{Mod}_{\mathcal{P}}$ and ${}_{\mathcal{P}}\mathbf{Mod}_{\mathcal{P}}$ are monoidal with respect to the level monoidal product $\otimes$.
- Let $(\mathcal{P}, \Delta)$ be a Hopf operad in $\mathbf{Ch}$. Then $\mathbf{F}_{\mathcal{P}}$ is the category such that
  - objects are chain coalgebras C endowed with a multiplicative right $\mathcal{P}$-module action $\mathcal{T}(\Omega C) \circ \mathcal{P} \to \mathcal{T}(\Omega C)$;
  - morphisms are chain coalgebra morphisms $f : C \to C'$ such that $\Omega f : \Omega C \to \Omega C'$ is a morphism of $\mathcal{P}$-coalgebras.

Diffracted right module maps I

Given:

- $(\mathcal{P}, \Delta)$, a Hopf operad in **Ch**
- C, a chain coalgebra,
- C', an object in $\mathbf{F}_{\mathcal{P}}$, with multiplicative action map $\psi' : \mathcal{T}(\Omega C') \circ \mathcal{P} \to \mathcal{T}(\Omega C')$,
- $(\mathcal{X}, \Delta, \rho)$, a level comonoid in the category of right $\mathcal{P}$-modules.

A transposed tensor morphism $\theta : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \to \mathcal{T}(C')$ is a diffracted right $\mathcal{P}$-module map if the following diagram commutes.

Diffracted right module maps II

$$
\begin{array}{ccc}
\mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X} \circ \mathcal{P}) & & \\
\Big\downarrow {\scriptstyle Id_{\mathcal{T}(C)} \underset{\mathcal{A}}{\circ} \Phi(\rho)} & \searrow^{\mathrm{Lin}\left[\psi'(\mathrm{Ind}\,\theta \circ Id_{\mathcal{P}})\right]} & \\
\mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) & \xrightarrow{\theta} & \mathcal{T}(C')
\end{array}
$$

The diagonal arrow in the diagram above is obtained by linearizing the composite

$$\mathcal{T}(\Omega C) \circ \mathcal{X} \circ \mathcal{P} \xrightarrow{\mathrm{Ind}(\theta) \circ Id_{\mathcal{P}}} \mathcal{T}(\Omega C') \circ \mathcal{P} \xrightarrow{\psi'} \mathcal{T}(\Omega C').$$

Diffracted balanced module maps I

Given:

- $(\mathcal{P}, \Delta)$, a Hopf operad in **Ch**
- C, an object in $\mathbf{F}_{\mathcal{P}}$, with multiplicative action map $\psi : \mathcal{T}(\Omega C) \circ \mathcal{P} \to \mathcal{T}(\Omega C)$,
- C', a chain coalgebra,
- $(\mathcal{X}, \Delta, \lambda)$, a level comonoid in the category of left $\mathcal{P}$-modules.

A transposed tensor morphism $\theta : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \to \mathcal{T}(C')$ is a diffracted balanced $\mathcal{P}$-module map if the following diagram commutes.

Diffracted balanced module maps II

$$\begin{array}{ccc} \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{P} \circ \mathcal{X}) & & \\ \Big\downarrow {\scriptstyle Id_{\mathcal{T}(C)} \underset{\mathcal{A}}{\circ} \Phi(\lambda)} & \searrow^{\mathrm{Lin}\left[\mathrm{Ind}\,\theta(\psi \circ Id_{\mathcal{X}})\right]} & \\ \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) & \xrightarrow{\theta} & \mathcal{T}(C') \end{array}$$

The diagonal arrow in the diagram above is obtained by linearizing the composite

$$\mathcal{T}(\Omega C) \circ \mathcal{P} \circ \mathcal{X} \xrightarrow{\psi \circ Id_{\mathcal{X}}} \mathcal{T}(\Omega C) \circ \mathcal{X} \xrightarrow{\mathrm{Ind}(\theta)} \mathcal{T}(\Omega C').$$

The solution of the problem

Theorem (H.-P.-S.)

Let $\theta : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{X}) \to \mathcal{T}(C')$ *be a transposed tensor map.*

- *Let* C' *be an object in* $\mathbf{F}_{\mathcal{P}}$. *Let* $(\mathcal{X}, \Delta, \rho)$ *be a level comonoid in the category of right* $\mathcal{P}$*-modules. Then* θ *is a diffracted right* $\mathcal{P}$*-module map if and only if* $\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \to \mathcal{T}(\Omega C')$ *is a right* $\mathcal{P}$*-module map.*

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  Then θ *is a diffracted right* $\mathcal{P}$*-module map if and only if* $\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \to \mathcal{T}(\Omega C')$ *is a right* $\mathcal{P}$*-module map.*

- *Let* C *be an object in* $\mathbf{F}_{\mathcal{P}}$*. Let* $(\mathcal{X}, \Delta, \lambda)$ *be a level comonoid in the category of left* $\mathcal{P}$*-modules.*
  Then θ *is a diffracted balanced* $\mathcal{P}$*-module map if and only if* $\mathrm{Ind}(\theta) : \mathcal{T}(\Omega C) \circ \mathcal{X} \to \mathcal{T}(\Omega C')$ *induces a morphism of symmetric sequences*
  $\widehat{\mathrm{Ind}(\theta)} : \mathcal{T}(\Omega C) \underset{\mathcal{P}}{\circ} \mathcal{X} \to \mathcal{T}(\Omega C')$*.*

Elements of the proof

Proof.

The proof follows easily from an Enriched Cobar Duality Theorem, which is expressed in terms of bundles of bicategories with connection.

This notion captures succintly the very high degree of naturality hidden in the induction and linearization transformations. □

Enriched existence theorems I

Theorem (H.-P.-S.)

Let $X : \mathbf{D} \to \mathbf{C}$ and $Y : \mathbf{D} \to \mathbf{F}_{\mathcal{P}}$ be functors, where $\mathbf{D}$ is a category admitting a set of models with respect to which X is free and Y is acyclic.

Let $\mathcal{M}$ be a semifree, level comonoid in the category of right $\mathcal{P}$-modules under $\mathcal{J}$.

Let $\tau : X \to UY$ be a natural transformation, where $U : \mathbf{F}_{\mathcal{P}} \to \mathbf{C}$ is the forgetful functor.

Then there is a natural, multiplicative right $\mathcal{P}$-module transformation

$$\theta : \mathcal{T}(\Omega X) \circ \mathcal{M} \to \mathcal{T}(\Omega Y)$$

extending $s^{-1}\tau$.

Enriched existence theorems II

Theorem (H.-P.-S.)

Let $X : \mathbf{D} \to \mathbf{F}_{\mathcal{P}}$ *and* $Y : \mathbf{D} \to \mathbf{C}$ *be functors, where* $\mathbf{D}$ *is a category admitting a set of models with respect to which* X *is free and* Y *is acyclic.*

Let $\mathcal{M}$ *be a semifree, level comonoid in the category of left* $\mathcal{P}$*-modules under* $\mathcal{J}$*.*

Let $\tau : UX \to Y$ *be a natural transformation, where* $U : \mathbf{F}_{\mathcal{P}} \to \mathbf{C}$ *is the forgetful functor.*

Then there is a natural, multiplicative transformation

$$\theta : \mathcal{T}(\Omega X) \underset{\mathcal{P}}{\circ} \mathcal{M} \to \mathcal{T}(\Omega Y)$$

extending $s^{-1}\tau$*.*

Proof of the enriched existence theorems

Proof.

Since $\mathcal{M}$ is semifree, there is a very nice differential filtration on $\Phi(\mathcal{M})$, which enables us to apply acyclic models methods to prove the existence of a diffracted right $\mathcal{P}$-module map (respectively, of a diffracted, balanced $\mathcal{P}$-module map)

$$\hat{\tau} : \mathcal{T}(X) \underset{\mathcal{A}}{\circ} \Phi(\mathcal{M}) \longrightarrow \mathcal{T}(Y).$$

Then set $\theta = \mathrm{Ind}(\hat{\tau})$. □

Existence of enriched DCSH-structure

Theorem (H.-P.-S.)

Let $X, Y : \mathbf{D} \to \mathbf{F}_{\mathcal{A}}$ be functors, where $\mathbf{D}$ is a category admitting a set of models with respect to which X is free and Y is acyclic.

Let $\tau : UX \to UY$ be a natural transformation of functors into **Coalg**.

Then there is a natural transformation $\theta : \Omega X \to \Omega Y$ of functors into **Alg** *such that $\theta(d)$ is naturally a DCSH-map for all $d \in \mathrm{Ob}\,\mathbf{D}$ and such that the composite*

$$X \xrightarrow{s^{-1}} s^{-1}X \hookrightarrow \Omega X \xrightarrow{\theta} \Omega Y \xrightarrow{\pi} s^{-1}Y \xrightarrow{s} Y$$

is exactly τ.

Proof of DCSH case

Proof.

$\mathcal{F} = \Phi(\mathcal{J})$ is semifree. $\square$

Application to double loops on a suspension

Recall $Sz_K : (\Omega C_* K, \psi_K) \xrightarrow{\simeq} (C_* GK, \Delta_{GK})$.

Theorem (H.-P.-S. II)

If $K = EL$ is a simplicial suspension, then

- *$\Omega^2 C_*(K)$ admits a natural, coassociative coproduct $\psi_{2,K}$, and*
- *$\Omega Sz_K : (\Omega^2 C_* K, \psi_{2,K}) \xrightarrow{\simeq} (\Omega C_* GK, \psi_{GK})$ is a chain algebra and DCSH equivalence.*

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Corollary

If $K = EL$ is a simplicial suspension, then

$$(\Omega^2 C_* K, \psi_{2,K}) \xrightarrow{\Omega Sz_K} (\Omega C_* GK, \psi_{GK}) \xrightarrow{Sz_{GK}} (C_* G^2 K, \Delta_{G^2 K})$$

is a chain algebra and DCSH equivalence.

Future work

- Generalize diffraction to all quadratic operads $\mathcal{Q}$ and then to their cofibrant replacements $\mathcal{Q}_\infty$.
- Further applications to algebraic topology, in particular to generalizing the result concerning double loops on a suspension.

Bundles of bicategories

Let $\mathbb{B}$ be a bicategory with exactly one 0-cell, and let $\mathbb{E}$ be any bicategory. Let $\Pi : \mathbb{E} \to \mathbb{B}$ be a bundle in the category of bicategories, i.e., a (strict) bicategory homomorphism.

For all $e, e' \in \mathbb{E}_0$,

$$\mathbb{E}_1(e, e') = \coprod_{b \in B_1} \mathbb{E}_1(e, e')_b,$$

where $\mathbb{E}_1(e, e')_b = \Pi_1^{-1}(b) \cap \mathbb{E}_1(e, e')$.

In terms of this decomposition,

$$\varphi \in \mathbb{E}_1(e, e')_b, \psi \in \mathbb{E}_1(e', e'')_{b'} \Rightarrow \psi \cdot \varphi \in \mathbb{E}_1(e, e'')_{b' \cdot b}.$$

op-Connections

An op-connection on a bicategory bundle $\Pi : \mathbb{E} \to \mathbb{B}$, where $\mathbb{B}$ has exactly one object, is of a family of functors natural in e and e'

$$\nabla = \{\nabla_{e,e'} : \mathbf{B}^{op} \to \mathbf{Cat} \mid e, e' \in \mathbb{E}_0\},$$

where $\mathbf{B}$ is the monoidal category corresponding to $\mathbb{B}$, such that

- $\nabla_{e,e'}(b) = \mathbb{E}_1(e, e')_b$ for all $b \in \mathrm{Ob}\,\mathbf{B}$, and therefore, for all $\alpha \in \mathbf{B}^{op}(b, b')$, there is a parallel transport functor

$$\nabla_{e,e'}(\alpha) : \mathbb{E}_1(e, e')_b \longrightarrow \mathbb{E}_1(e, e')_{b'};$$

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$$\nabla_{e,e'}(\alpha) : \mathbb{E}_1(e, e')_b \longrightarrow \mathbb{E}_1(e, e')_{b'};$$

- for all $\alpha \in \mathbf{B}^{op}(b, b')$, $\overline{\alpha} \in \mathbf{B}^{op}(\bar{b}, \bar{b}')$ and $\varphi \in \mathbb{E}_1(e, e')_b$, $\overline{\varphi} \in \mathbb{E}_1(e', e'')_{\bar{b}}$,

$$\nabla_{e,e''}(\overline{\alpha} \otimes \alpha)(\overline{\varphi} \cdot \varphi) = \nabla_{e',e''}(\overline{\alpha})(\overline{\varphi}) \cdot \nabla_{e,e'}(\alpha)(\varphi).$$

Morphisms of bundles with op-connections

A morphism from $(\Pi : \mathbb{E} \to \mathbb{B}, \nabla)$ to $(\Pi' : \mathbb{E}' \to \mathbb{B}', \nabla')$ consists of a pair of bicategory homomorphisms $\Gamma : \mathbb{E} \to \mathbb{E}'$ and $\Lambda : \mathbb{B} \to \mathbb{B}'$ such that

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\Gamma} & \mathbb{E}' \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi'} \\ \mathbb{B} & \xrightarrow{\Lambda} & \mathbb{B}' \end{array}$$

commutes.

Also, for all $e, e' \in \mathbb{E}_0$, $\varphi \in \mathbb{E}(e, e')_b$, $\alpha \in \mathbf{B}^{op}(b, b')$

$$\Gamma_1\big(\nabla_{e,e'}(\alpha)(\varphi)\big) = \nabla'_{\Gamma_0(e),\Gamma_0(e')}\big(L(\alpha)\big)\big(\Gamma_1(\varphi)\big),$$

where $L : (\mathbf{B}, \otimes) \to (\mathbf{B}', \otimes)$ is the strict monoidal functor associated to Λ.

Enriched cobar duality

Theorem

Induction and linearization give rise to mutually inverse isomorphisms of bicategory bundles with connection

$$(\Pi^{\Omega} : \mathbb{DC}^{\Omega} \to \mathbb{CM}, \nabla^{\Omega}) \underset{\cong}{\overset{\cong}{\leftrightarrows}} (\Pi^{\Phi} : \mathbb{DC}^{\Phi} \to \mathbb{CM}, \nabla^{\Phi}),$$

where

- $\mathbb{CM}$ *is the bicategory corresponding to* **Comon**$_{\otimes}$*,*
- $\mathbb{DC}^{\Omega}$ *generalizes* **DCSH***, and*
- $\mathbb{DC}^{\Phi}$ *generalizes* $(\mathcal{A}, \mathcal{F})$-**Coalg***.*