Lax-algebraic theories and closed objects

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A ***lax-algebraic theory*** $\mathscr{T}$ is a triple $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of

a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and

a map $\xi : T\mathsf{V} \to \mathsf{V}$

such that

(M$_\mathrm{e}$) $1_{\mathsf{V}} \leq \xi \cdot e_{\mathsf{V}}$, (M$_\mathrm{m}$) $\xi \cdot T\xi \leq \xi \cdot m_{\mathsf{V}}$,

(Q$_\otimes$)
$$\begin{array}{ccc} T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T(\otimes)} & T\mathsf{V} \\ \downarrow & \leq & \downarrow \xi \\ \mathsf{V} \times \mathsf{V} & \xrightarrow[\otimes]{} & \mathsf{V}, \end{array}$$

(Q$_\mathrm{k}$)
$$\begin{array}{ccc} T1 & \xrightarrow{Tk} & T\mathsf{V} \\ ! \downarrow & \leq & \downarrow \xi \\ 1 & \xrightarrow[k]{} & \mathsf{V}, \end{array}$$

(Q$_\bigvee$) $(\xi_X)_X : P_{\mathsf{V}} \to P_{\mathsf{V}} T$ is a natural transformation.

Examples.

(a). $\mathscr{I}_{\mathsf{V}} = (\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$ is a strict lax-algebraic theory.

(b). Let $\mathbb{T} = (T, e, m)$ be a monad where T is taut and let V be a (ccd)-quantale. Then $\mathscr{T}_{\mathsf{V}} = (\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ is a lax-algebraic theory, where

$$\xi_{\mathsf{V}} : T\mathsf{V} \to \mathsf{V}, \;\; \mathfrak{x} \mapsto \bigvee \{v \in \mathsf{V} \mid \mathfrak{x} \in T(\uparrow v)\}.$$

(c). $\mathscr{L}_{\mathsf{V}}^{\otimes} = (\mathbb{L}, \mathsf{V}, \xi_{\otimes})$ is a strict lax-algebraic theory for each quantale V, where

$$\xi_{\otimes} : L\mathsf{V} \to \mathsf{V}.$$
$$(v_1, \ldots, v_n) \mapsto v_1 \otimes \ldots \otimes v_n$$
$$() \mapsto k$$

The bicategory $\mathsf{V}\text{-}\mathsf{Mat}$:

- objects: sets X, Y,...
- morphism: V-matrices $r : X \times Y \to \mathsf{V}$,
- composition: $s \cdot r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$

We extent $T : \mathsf{Set} \to \mathsf{Set}$ to $T_\xi : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ by putting

$$T_\xi r : TX \times TY \to \mathsf{V}.$$

$$(\mathfrak{x}, \mathfrak{y}) \mapsto \bigvee_{\substack{\mathfrak{w} \in T(X \times Y): \\ T\pi_X(\mathfrak{w}) = \mathfrak{x}, \\ T\pi_Y(\mathfrak{w}) = \mathfrak{y}}} \xi \cdot Tr(\mathfrak{w})$$

Here

$$T(X \times Y) \xrightarrow{Tr} T\mathsf{V} \xrightarrow{\xi} \mathsf{V}.$$

The following statements hold.

(a). For each V-matrix $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$.

(b). For each function $f : X \to Y$, $Tf \leq T_\xi f$ and $Tf^\circ \leq T_\xi f^\circ$.

(c). $T_\xi s \cdot T_\xi r \leq T_\xi(s \cdot r)$ provided that T satisfies (BC), and
$T_\xi s \cdot T_\xi r \geq T_\xi(s \cdot r)$ provided that $(\mathrm{Q}^{=}_{\otimes})$ holds.

(d). The natural transformations e and m become op-lax, that is, for every V-matrix $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ we have the inequalities:

$$e_Y \cdot r \leq T_\xi r \cdot e_X, \qquad m_Y \cdot T_\xi T_\xi r \leq T_\xi r \cdot m_X.$$

$$\begin{array}{ccc} X & \xrightarrow{e_X} & T_\xi X \\ {\scriptstyle r}\downarrow & \leq & \downarrow{\scriptstyle T_\xi r} \\ Y & \xrightarrow[e_Y]{} & T_\xi Y \end{array} \qquad\qquad \begin{array}{ccc} T_\xi T_\xi X & \xrightarrow{m_X} & T_\xi X \\ {\scriptstyle T_\xi T_\xi r}\downarrow & \leq & \downarrow{\scriptstyle T_\xi r} \\ T_\xi T_\xi Y & \xrightarrow[m_Y]{} & T_\xi Y \end{array}$$

Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a lax-algebraic theory.

- A ***$\mathscr{T}$-algebra ($\mathscr{T}$-category)*** is a pair $(X, a : TX \,{\to\!\!\!\!\!\!+}\; X)$ s. t.

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & \underset{1_X}{\diagdown\!\!\diagdown} \;\; \leq & \big\downarrow\!\!\!+ a \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ T_\xi a \big\downarrow\!\!\!+ & \leq & \big\downarrow\!\!\!+ a \\ TX & \xrightarrow[a]{\quad+\quad} & X. \end{array}$$

$$\boxed{k \to a(\dot{x}, x)} \qquad\qquad \boxed{T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \to a(m_X(\mathfrak{X}), x)}$$

- A map $f : X \to Y$ between $\mathscr{T}$-algebras (X, a) and (Y, b) is a ***lax homomorphism ($\mathscr{T}$-functor)*** if

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \big\downarrow\!\!\!+ & \leq & \big\downarrow\!\!\!+ b \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \boxed{a(\mathfrak{x}, x) \to b(Tf(\mathfrak{x}), f(x)).}$$

• The resulting category of $\mathscr{T}$-algebras and lax homomorphisms we denote by $\mathscr{T}$-Alg.

Examples.

(a). For each quantale V, $\mathscr{I}_{\mathsf{V}}$-$\mathsf{Alg} = \mathsf{V}$-$\mathsf{Cat}$.
In particular, $\mathscr{I}_{\mathsf{2}}$-$\mathsf{Alg} \cong \mathsf{Ord}$ and $\mathscr{I}_{\mathsf{P}_+}$-$\mathsf{Alg} \cong \mathsf{Met}$.

(b). $\mathscr{U}_{\mathsf{2}}$-$\mathsf{Alg} \cong \mathsf{Top}$.

(c). $\mathscr{U}_{\mathsf{P}_+}$-$\mathsf{Alg} \cong \mathsf{Ap}$.

(d). $\mathscr{L}_{\mathsf{V}}^{\otimes}$-$\mathsf{Alg} \cong \mathsf{V}$-$\mathsf{MultiCat}$.

Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ and $\mathscr{T}' = (\mathbb{T}', \mathsf{V}', \xi')$ be lax-algebraic theories.

• ***A morphism*** $(j, \varphi) : \mathscr{T}' \to \mathscr{T}$ ***of lax-algebraic theories*** is a pair (j, φ) consisting of

a monad morphism $j : \mathbb{T}' \to \mathbb{T}$ and

a lax homomorphism of quantales $\varphi : \mathsf{V} \to \mathsf{V}'$

such that $\boxed{\xi' \cdot T'\varphi \leq \varphi \cdot \xi \cdot j_{\mathsf{V}}}$.

$$\begin{array}{ccc} T'\mathsf{V} & \xrightarrow{\;j_{\mathsf{V}}\;} & T\mathsf{V} \\ {\scriptstyle T'\varphi}\downarrow & & \\ T'\mathsf{V}' & \leq & \downarrow{\scriptstyle \xi} \\ {\scriptstyle \xi'}\downarrow & & \\ \mathsf{V}' & \xleftarrow[\;\varphi\;]{} & \mathsf{V} \end{array}$$

From now on we consider a strict lax-algebraic theory $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ where $\mathbb{T}$ satisfies (BC).

Examples.

(a). The identity theory $\mathscr{I}_{\mathsf{V}}$, for each quantale V.

(b). For each quantale V, the theory $\mathscr{L}_{\mathsf{V}}^{\otimes} = (\mathbb{L}, \mathsf{V}, \xi_{\otimes})$.

(c). Any lax-algebraic theory $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ with a (BC)-monad $\mathbb{T}$, $\otimes = \wedge$ and ξ a Eilenberg-Moore algebra.

(d). The theory $\mathscr{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$.

Then

- V becomes a $\mathscr{T}$-algebra $(\mathsf{V}, \hom_\xi)$ where $\hom_\xi = \hom \cdot \xi$, that is,

$$\hom_\xi(\mathfrak{v}, v) = \hom(\xi(\mathfrak{v}), v).$$

- the tensor product $\otimes$ on V can be transported to $\mathscr{T}$-Alg by putting $(X, a) \otimes (Y, b) = (X \times Y, c)$ where

$$c(\mathfrak{w}, (x, y)) = a(\mathfrak{x}, x) \otimes b(\mathfrak{y}, y).$$

When $X \otimes \_$ has a right adjoint $\_\_^X$?

Note that

$$\frac{1 \to Y^X}{X \otimes 1 \to Y}$$

Hence we consider

$$\{f : \hat{X} \to Y \mid f \text{ is a lax homomorphism}\},$$

where

$$\hat{a}(\mathfrak{x}, x) = \begin{cases} a(\mathfrak{x}, x) & \text{if } T!(\mathfrak{x}) = e_1(\star), \\ \bot & \text{else;} \end{cases}$$

and

$$\boxed{d(\mathfrak{p}, h) = \bigwedge_{\substack{\mathfrak{q} \in T(Y^X \times X), x \in X \\ \mathfrak{q} \mapsto \mathfrak{p}}} \hom(a(T\pi_X(\mathfrak{q}), x), b(T\mathrm{ev}(\mathfrak{q}), h(x))).}$$

Letv $X = (X, a)$ be a $\mathscr{T}$-algebra.

- Assume that $a \cdot T_\xi a = a \cdot m_X$. Then d is transitive.
- Assume that the structure d on V^X is transitive. Then $a \cdot T_\xi a = a \cdot m_X$.
- Each $\mathbb{T}$-algebra is closed in $\mathscr{T}$-Alg.
- Each V-category is closed in $\mathscr{T}$-Alg provided that $Te \cdot e = m^\circ \cdot e$.

The following assertions hold.

- $\bigwedge : \mathsf{V}^I \to \mathsf{V}$ is a lax homomorphism.
- $\hom(v, \_) : \mathsf{V} \to \mathsf{V}$ is a lax homomorphism for each $v \in \mathsf{V}$.
- $v \otimes \_ : \mathsf{V} \to \mathsf{V}$ is a lax homomorphism for each $v \in \mathsf{V}$ which satisfies

$$\begin{array}{ccc} T1 & \xrightarrow{Tv} & T\mathsf{V} \\ {\scriptstyle !}\big\downarrow & \geq & \big\downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[v]{} & \mathsf{V}. \end{array}$$

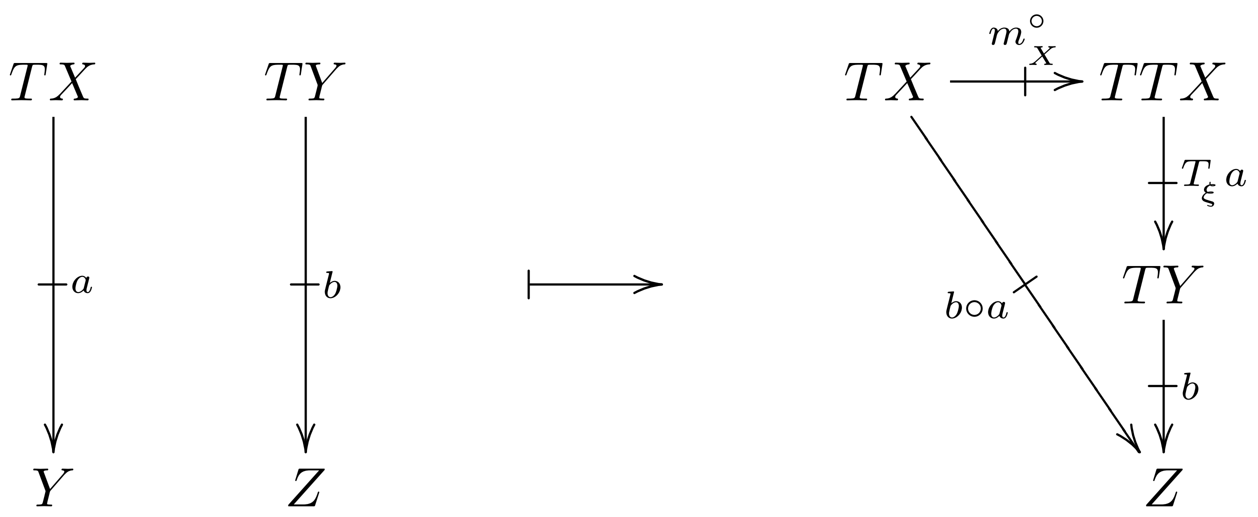
- For each $\mathbb{T}$-algebra I, $\bigvee : \mathsf{V}^I \to \mathsf{V}$ is a lax homomorphism.

$\mathscr{T}$-Kleisli.

objects: sets X, Y, ...

morphism: V-matrices $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

composition: $b \circ a := b \cdot T_{\xi} a \cdot m_X^{\circ}$,



Then $e_X^\circ : TX \to\!\!\!\!\!\!\mapsto X$ is a lax identity for "$\circ$", that is

$$\boxed{a \circ e_X^\circ = a} \qquad \text{and} \qquad \boxed{e_X^\circ \circ a \geq a}.$$

Moreover, $\boxed{c \circ (b \circ a) = (c \circ b) \circ a}$.

$(X, a : TX \to\!\!\!\!\!\!\mapsto X)$ is a $\mathscr{T}$-algebra iff $e_X^\circ \leq a$ and $a \circ a \leq a$.

Example: $\mathscr{U}_2$

- e_X° is also a left unit (precisely) if we restrict ourself to those $a : UX \to\!\!\!\!\!\!\mapsto Y$ where $\{\mathfrak{x} \in UX \mid a(\mathfrak{x}, y) = \text{true}\}$ is closed in UX.
- This restriction of $\mathscr{U}_2$-Kleisli is 2-equivalent to CSet (where a morphism from X to Y is a finitely additive map $c : PX \to PY$).

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathscr{T}$-algebras.

• A $(\mathbb{T}, \mathsf{V})$-***bimodule*** $\psi : (X, a) \multimap (Y, b)$ is a matrix $\psi : TX \nrightarrow Y$ such that $\psi \circ a \leq \psi$ and $b \circ \psi \leq \psi$.

• For $(\mathbb{T}, \mathsf{V})$-categories (X, a) and (Y, b), and a V-matrix $\psi : TX \nrightarrow Y$, the following assertions are equivalent.

(a). $\psi : (X, a) \multimap (Y, b)$ is a $(\mathbb{T}, \mathsf{V})$-bimodule.

(b). Both $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T}, \mathsf{V})$-functors.