The category of realizability toposes

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Introduction: the other side of the fence...

Enviable aspects of Grothendieck toposes:

- We know what a Grothendieck topos is.
- Characterizations (sheaves on a site, Giraud's theorem).
- 2-category of Grothendieck toposes has various good closure properties.
- There are nice representation theorems.

This side of the fence...

- Interesting examples: Effective topos, toposes for various other types of realizability.

- Constructions and presentations of such toposes via indexed categories, completions.

1. Can we abstractly characterize/define realizability toposes?
2. How can we understand morphisms of realizability toposes?
3. Are there useful representation theorems?
4. What constructions can we perform on realizability toposes?

Basic combinatorial objects.

We consider systems $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$, where Σ is a set, $\leq$ is a partial ordering of Σ, and $\mathcal{F}_\Sigma$ is a class of partial monotone endofunctions on Σ.

Such a system is called a *basic combinatorial object* (BCO for short) if the class $\mathcal{F}_\Sigma$ has the following properties:

- For $f \in \mathcal{F}_\Sigma$, $dom(f)$ is downward closed
- $1_\Sigma \in \mathcal{F}_\Sigma$
- $f, g \in \mathcal{F}_\Sigma \Rightarrow fg \in \mathcal{F}_\Sigma$.

We think of the functions $f \in \mathcal{F}_\Sigma$ as the *computable* or *realizable* functions on Σ.

Morphisms of BCOs.

Given $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ and $\Theta = (\Theta, \leq, \mathcal{F}_\Theta)$, a morphism $\phi : \Sigma \to \Theta$ is a function on the underlying sets such that

- there exists $u \in \mathcal{F}_\Theta$ such that for all $a \leq a'$ in Σ we have $u(\phi(a)) \leq \phi(a')$;
- for all $f \in \mathcal{F}_\Sigma$ there exists $g \in \mathcal{F}_\Theta$ with $g\phi(a) \leq \phi(f(a))$ for all $a \in dom(f)$.

The following diagram serves as a heuristics for the second condition:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \phi\ } & \Theta \\ {\scriptstyle \forall f \in \mathcal{F}_\Sigma}\downarrow & \geq & \vdots\downarrow{\scriptstyle \exists g \in \mathcal{F}_\Theta} \\ \Sigma & \xrightarrow{\ \phi\ } & \Theta. \end{array}$$

BCOs and morphisms form a category **BCO**. This category is in fact pre-order enriched: for two parallel morphisms $\phi, \psi : \Sigma \to \Theta$, we define

$$\phi \vdash \psi \Leftrightarrow \exists g \in \mathcal{F}_\Theta \forall a \in \Sigma . g\phi(a) \leq \psi(a).$$

Note: this is in general not a pointwise ordering.

Definition. A BCO Σ is called *cartesian* if both maps $\Sigma \to \Sigma \times \Sigma$ and $\Sigma \to 1$ have right adjoints, which we then denote by $\wedge : \Sigma \times \Sigma \to \Sigma$ and $\top : 1 \to \Sigma$. A morphism between cartesian BCOs is called cartesian if it preserves the cartesian structure up to isomorphism.

The sub-2-category on the cartesian objects and morphisms will be denoted by $\mathbf{BCO_{cart}}$.

Examples.

1. Every poset can be viewed as a BCO: the only computable function will be the identity. This gives a full 2-embedding of the 2-category of posets into **BCO**. It restricts to an embedding of meet-semilattices into $\mathbf{BCO_{cart}}$.
2. Consider the natural numbers $\mathbb{N}$ with the discrete ordering. Declare each partial recursive function to be computable. This gives in fact a cartesian BCO, using the recursion-theoretic pairing $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
3. Every PCA is a cartesian BCO, see next slides.

Partial Combinatory Algebras.

Partial applicative structures. Let A be a set, endowed with a partial application

$$\bullet : A \times A \rightharpoonup A.$$

Notation. Write abc for $(a \bullet b) \bullet c$; write $ab{\downarrow}$ for $(a, b) \in dom(\bullet)$.

Every element $b \in A$ is thought of as representing a function, namely the function $a \mapsto b \bullet a$.

More generally, a (partial) function $f : A^{n+1} \rightharpoonup A$ is said to be *represented* by an element $b \in A$ when for all $a_1, \ldots, a_{n+1} \in A$:

- $b \bullet a_1 \cdots a_{n+1} \simeq f(a_1, \ldots, a_{n+1})$
- $b \bullet a_1 \cdots a_n{\downarrow}$.

Fix a partial applicative structure $(A, \bullet)$. A *term* over A is an expression built from elements of A, variables and brackets using $\bullet$.

E.g., $(a \bullet x_2) \bullet (x_3 \bullet x_1)$, x_2 and $b \bullet b$ are terms.

A term t with $FV(t) \subset \{x_1, \dots, x_n\}$ may be viewed as a polynomial function $A^n \rightharpoonup A$.

Definition. We say that $A = (A, \bullet)$ is a PCA when every term is representable by an element of A.

- write $\lambda \overrightarrow{x}.t$ for the element representing t
- one can define a representable pairing operation $\langle -, - \rangle : A \times A \to A$
- every PCA contains a copy of $\mathbb{N}$ such that every recursive function is representable.

Examples (continued).

Fact: there is a full 2-embedding of PCAs into the category $\mathbf{BCO_{cart}}$.

This suggests that (cartesian) BCOs comprise a spectrum of objects, with on one extreme lattices (purely order-theoretic/spatial) and on the other extreme PCAs (purely combinatorial).

What's in between?

- Ordered PCAs (underlying set is partially ordered, representability conditions now hold up to inequality). Given a PCA A, the non-empty subsets from an ordered PCA via $U \bullet V \simeq \{uv | u \in U, v \in V\}$.
- Given a PCA A and a full sub-PCA $B \subseteq A$ one can consider relative computability: the computable functions on A are those of the form $b \bullet -$ for $b \in B$.
- Combine the above two.

From BCOs to logic.

Fix a BCO Σ. For an arbitrary set X, we define a preorder on the set $[X, \Sigma]$ as

$$\alpha \vdash_X \beta \Leftrightarrow \exists f \in \mathcal{F}_\Sigma . \forall x \in X . f(\alpha(x)) \leq \beta(x).$$

- Σ is a collection of truth-values

- X is a type

- $\alpha, \beta : X \to \Sigma$ are predicates with a free variable of type X

$X \mapsto [X, \Sigma]$ defines a **Set**-indexed preorder, denoted $[-, \Sigma]$.

This defines a 2-functor **BCO** $\to$ **Set**-indexed preorders. This is a 2-embedding.

Example. If Σ arises from the PCA $\mathbb{N}$, then the preorder in the fibre over X is:

$$\alpha \vdash_X \beta \Leftrightarrow \exists n . \forall x . n \bullet \alpha(x) = \beta(x).$$

Look for correspondence:

$$\text{properties of } \Sigma \leftrightarrow \text{properties of } [-,\Sigma]$$

For example:

Σ is cartesian $\Leftrightarrow [-,\Sigma]$ has indexed finite limits.

Less trivial: when does $[-,\Sigma]$ have existential quantification? Consider the following construction: for a BCO Σ, put

$$\mathcal{D}(\Sigma) = \{U \subseteq \Sigma | U \text{ is downward closed}\}.$$

This is ordered by inclusion, and a partial monotone function $F : \mathcal{D}(\Sigma) \rightharpoonup \mathcal{D}(\Sigma)$ is defined to be computable if there is an $f \in \mathcal{F}_\Sigma$ such that

$$U \in dom(F) \Rightarrow \forall a \in U.\ f(a)\downarrow \ \&\ f(a) \in F(U).$$

Downset monad.

Fact. The functor $\mathcal{D}$ is a KZ-monad on **BCO**.

Proposition. The following are equivalent:

- The indexed preorder $[-, \Sigma]$ has existential quantification
- The BCO Σ is a pseudo-algebra for the monad $\mathcal{D}$.

Remarks.

1) Because $\mathcal{D}$ is KZ, a pseudo-algebra structure is necessarily unique up to isomorphism.

2) Applying $\mathcal{D}$ to the example $\Sigma = \mathbb{N}$ gives the Effective tripos.

3) There is a variation: replace $\mathcal{D}$ by $\mathcal{D}_i$, *inhabited downsets*. The above result then is true when we restrict to quantification along surjective maps.

Tripos characterizations.

From now we work in the category $\mathbf{BCO_{cart}}$. Define

$$TV(\Sigma) = \{a \in \Sigma | \top \vdash a\}.$$

The set $TV(\Sigma)$ is upwards closed, and is closed under conjunction. Its elements are called *designated truth-values.*

Theorem (Free case). The following are equivalent for a cartesian BCO Σ:

- $[-, \mathcal{D}(\Sigma)]$ is a tripos;
- There is an ordered PCA structure on Σ, the filter $TV(\Sigma)$ is a sub-ordered PCA, and the BCO structure on Σ arises in the canonical way from this data.

These are free triposes: existential quantification has been freely added.

Tripos characterizations (continued).

The general case is the following:

Theorem. The following are equivalent for a cartesian pseudo-algebra Σ:

- $[-, \Sigma]$ is a tripos;
- There is an ordered PCA structure on Σ, the filter $TV(\Sigma)$ is a sub-ordered PCA, and the BCO structure on Σ arises in the canonical way from this data. *In addition*, the algebra structure map should preserve application in the first variable (up to isomorphism).

This covers a number of non-free triposes, such as the tripos for modified realizability and the dialectica tripos.

Some side results.

Theorem. The operation $\Sigma \mapsto \mathcal{D}_i(\Sigma)$ preserves the property of being a tripos.
("Extensionalizing" a tripos.)

This gives rise to hierarchies of triposes.

Theorem. The topos corresponding to a free tripos $[-, \mathcal{D}(\Sigma)]$ is an exact completion, namely of the total category of the indexed category $[-, \Sigma]$.

(If we don't work over Set but over a topos which doesn't satisfy AC, then replace exact completion by relative exact completion.)

Geometric morphisms.

Definition (informal). A morphism of BCOs $\phi : \Sigma \to \Theta$ is *computationally dense* if

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi} & \Theta \\ {\scriptstyle \exists f \in \mathcal{F}_\Sigma}\ \vdots & & \vdash \ \downarrow {\scriptstyle \forall g \in \mathcal{F}_\Theta} \\ \Sigma & \xrightarrow{\phi} & \Theta. \end{array}$$

Theorem. For $\phi : \Sigma \to \Theta$, the following are equivalent:

- ϕ is computationally dense
- $\mathcal{D}(\phi) : \mathcal{D}(\Sigma) \to \mathcal{D}(\Theta)$ has a right adjoint
- $[-, \mathcal{D}(\phi)] : [-, \mathcal{D}(\Sigma)] \to [-, \mathcal{D}(\Theta)]$ has a right adjoint

Geometric morphisms, continued.

Theorem. For a $\mathcal{D}$-algebra Σ, the following are equivalent:

- ϕ is computationally dense
- ϕ has a right adjoint

Theorem. There is a natural isomorphism

$$\mathbf{BCO_d}(\Sigma, \mathcal{D}\Theta) \cong Geom(\mathcal{D}\Theta, \mathcal{D}\Sigma).$$

This gives a complete characterization of triposes and geometric morphisms arising from BCOs. (Also works on 2-cells.)

Example: Consider, for an algebra Σ, the map $\top : 1 \to \Sigma$. Density of this map is equivalent to $[-, \Sigma]$ being a localic tripos (i.e. Σ is equivalent to a locale).

Example: Consider $\mathbb{N} \hookrightarrow \mathbb{N}_A$, where A is an oracle.

Application.

Let $\phi : \Sigma \to \Theta$ be a morphism of cartesian BCOs.

Build a new BCO $\Sigma \ltimes \Theta$ as follows. The underlying set of $\Sigma \ltimes \Theta$ is simply $\Sigma \times \Theta$, ordered coordinatewise. Define the class of computable functions to be those of the form

$$(x, y) \mapsto (fx, g(\phi(x) \wedge y))$$

where $f \in \mathcal{F}_\Sigma, g \in \mathcal{F}_\Theta$.

This defines a comma square:

$$\begin{array}{c} (x, y) \\ \big\downarrow \\ \phi(x) \wedge y \end{array} \qquad\qquad \begin{array}{ccc} \Sigma \ltimes \Theta & \longrightarrow & \Sigma \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \Theta & \xrightarrow[1]{} & \Theta \end{array}$$

Proposition. Let $\phi : \Sigma \to \Theta$ be a cartesian morphism.

- If ϕ is a map of $\mathcal{D}$-algebras, then $\Sigma \ltimes \Theta$ is a $\mathcal{D}$-algebra.
- If ϕ is a map of (ordered) PCAs (with filters) then $\Sigma \ltimes \Theta$ is an (ordered) PCA (with filter).
- If ϕ is a map of triposes then $\Sigma \ltimes \Theta$ is a tripos.
- The projection $\Sigma \ltimes \Theta \to \Theta$ is computationally dense.

Now let $\phi : \Sigma \to \Theta$ be a map of ordered PCAs. In the realizability topos $\mathbf{RT}(\Theta)$ over Θ, this exhibits Σ as an *internal projective PCA*. Thus, we can build the realizability topos over this internal PCA:

$$\mathbf{Set} \to \mathbf{RT}(\Theta) \to \mathbf{RT}_{\mathbf{RT}(\Theta)}(\Sigma).$$

By Pitts' iteration theorem, the resulting topos should come from a tripos over **Set**.

Theorem. There is a natural equivalence of realizability toposes

$$\mathbf{RT}_{\mathbf{RT}(\Theta)}(\Sigma) \simeq \mathbf{RT}(\Sigma \ltimes \Theta).$$

The geometric morphism $\mathbf{RT}(\Theta) \to \mathbf{RT}(\Sigma \ltimes \Theta)$ corresponds to the computationally dense projection $\Sigma \ltimes \Theta \to \Theta$.