Smoothness In Zariski Categories

A Proposed Definition and a Few Easy Results.

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Outline

- Context

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Context

In 1992, Yves Diers published his inspirational book “Categories of Commutative Algebras”. As he himself says in the Introduction to “Categories of Commutative Algebras” his work stands in a line of advances on the problem of classifying categories. He specifically wants to understand categories which can are very similar to the category of commutative rings with identity.

Z. Luo

Was one of those who were inspired. In his work, available at www.geometry.net/cg, Z. Luo built upon Diers work in the dual situation. He argued that this side was the geometric, , and Diers was the algebraic.

Smoothness

Smoothness, I think, is traditionally understood as a geometric property

So, In Spite of the Title of My Talk

I'd like to talk about Smoothness as a geometric property and, so, for the most part, use Luo's language.

Equivalence

Dier’s Zariski category is roughly dual to Luo’s left coherent ’analytic geometry’.

Among other properties, it’s possessed of a strict initial object

a category in which limits commute with finite sums

locally disjunctable, reducible, and perfect.

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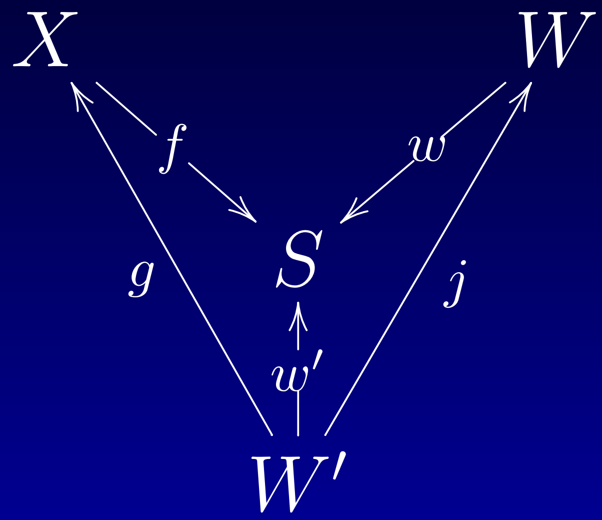
- Net - or unramified
- Lisse - or smooth
- Etale - or étale (slack....)

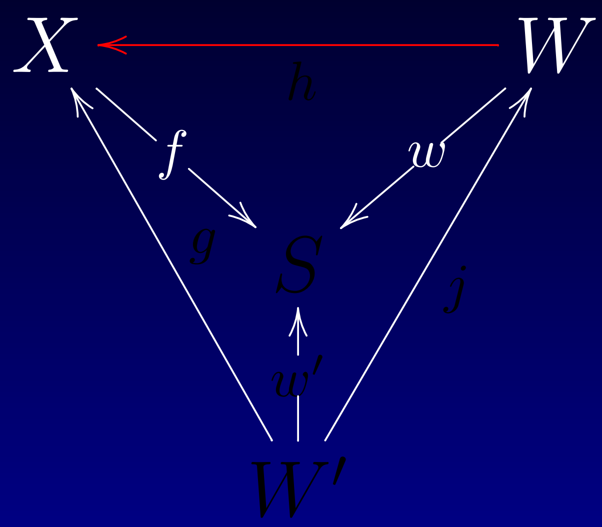
Grothendieck's Definition

Consider the commutative diagram



with j a strong unipotent mono.

If there exists



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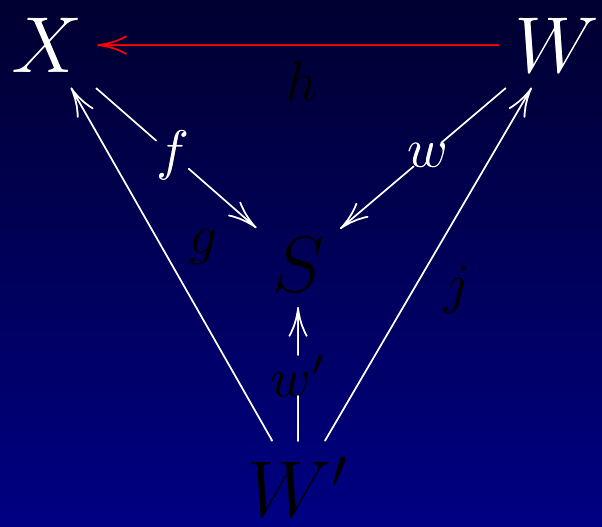

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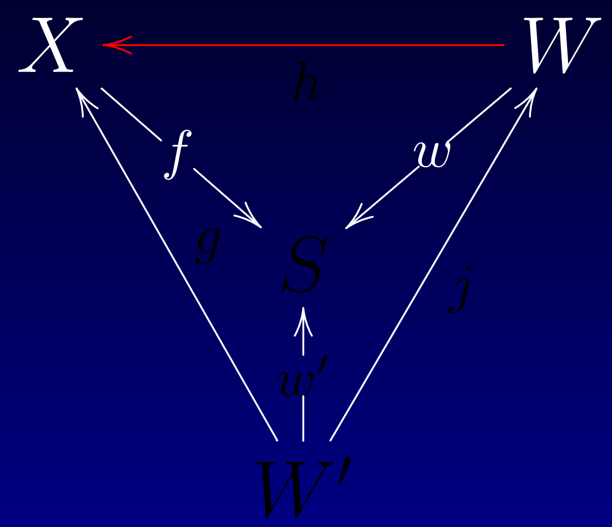

- at most one such h, then f is net
- at least one such h, then f is smooth
- exactly one such h, then f is étale

Definition

A strong unipotent (Luo's definition) mono is approximately the equivalent of a closed morphism in algebraic geometry induced by a morphism with unipotent kernel. Luo defines a unipotent morphism to be one which has no non-zero pullback isomorphic to zero. Geometrically, its image is not disjoint with anything.

Clearly

Sticking with Grothendieck's definition, if a category were to have no proper strong monic unipotents, every arrow would be étale.

Dier's Problem:

These three types of morphisms play an exceptionally important role in algebraic geometry. However, so-called reduced categories - that is categories without nilpotents satisfy the axioms for a Zariski category.

Subtext

Can we develop a more widely meaningful definition of these concepts, so that perhaps this axiomatic and categorical approach provide us with fresh insight into these concepts?

Dier's Solution - for net and étale

(In Luo's language)

- A morphism $f : X \longrightarrow S$ is net if the corresponding diagonal

$$\Delta : X \rightarrow X \times_S X$$

is a local isomorphism.

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- A morphism $f : X \to S$ is net if the corresponding diagonal

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is a local isomorphism.

- $f : X \to S$ is étale if it is net and coflat.

This is a very nice approach. However,

The definition of Smooth is missing

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- We could try to adapt the work of Anders Kock
- We could try to understand what ’locally linear’ means given Diers and Luo’s frame work.

Philosophy

In Calculus we teach students that to say that a function is differentiable is to say that in a sufficiently small neighborhood, the function is linear.

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- It wasn't clear to me how to define 'linear' in categorical terms

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- Linked to Dier’s definitions of net and étale
- Geometrically consistent with how we understand differentiability in more familiar settings, like Calculus
- As in algebraic geometry we want $S[T] \to S$ to be smooth

My Proposal

An arrow $f : X \to S$ will be called smooth if it is coflat and if there exists a unipotent analytic cover $\{U_i\}_{i \in I}$ of X so that for all $i \in I$ there exists $r > 0$ so that

$$\begin{array}{ccc} U_i & \xrightarrow{u_i} & X \\ \downarrow & & \downarrow f \\ \prod_r S' & \xrightarrow[\phi]{} & S \end{array}$$

commutes where $\prod_r S'$ is the product of r cogenerators and the arrow $U_i \to \prod_r S'$ is étale.

Unfortunately,

It turns out it was also Grothendieck's idea first. In fact, in Grothendieck's Universe, the two are equivalent as long as f is finitely presented

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- Lisse is a local property
- Lisse is not reflected by pullbacks
- Étale implies lisse
- Lisse doesn't imply étale
- $\overline{S} \longrightarrow S$ is lisse where $\overline{S}$ is a cogenerator

Questions

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- Develop a nice, useful definition of perfect field. Regularity vs. Smoothness...
- Develop classification of maps which are not smooth and catalog the effect that blow-ups have on such.

A Hopeful Sign

LWSR

Suppose $V \to W$ is a map of varieties. Then there exists blow-ups $\tilde{V} \to V$ and $\tilde{W} \to W$ so that the diagram

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \tilde{W} & \longrightarrow & W \end{array}$$

commutes and the canonical map $\tilde{V} \to \tilde{W}$ is flat.

A proof of LSWR would complete the proof of desingularization of 3 dimensional varieties and give a big leg up on the desingularization of those in higher dimension.