The Information Flow Framework: New Architecture

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"Philosophy cannot become scientifically healthy without an immense technical vocabulary. We can hardly imagine our great-grandsons turning over the leaves of this dictionary without amusement over the paucity of words with which their grandsires attempted to handle metaphysics and logic. Long before that day, it will have become indispensably requisite, too, that each of these terms should be confined to a single meaning, which, however broad, must be free from all vagueness. This will involve a revolution in terminology; for in its present condition a philosophical thought of any precision can seldom be expressed without lengthy explanations."

Charles Sanders Peirce, Collected Papers 8:169.

Contents

1 Introduction

What is the IFF?

The Information Flow Framework (IFF)[1] is a descriptive category metatheory currently under development that provides an important practical application of category theory to knowledge representation, knowledge maintenance and the semantic web.

Why is the IFF needed?

To quote David Whitten in the common logic forum:

> *"We (in the knowledge sciences, technologies and industries) are now in a situation where we don't have a common vocabulary at all. We now can't (really) evaluate if two systems are the same or not, because we don't have the formalized packages which express their distinctions. We don't have a computational architecture which is rich enough, and formally defined enough that we can (formally) point out the differences in two different approaches to the same problem."*

- The Information Flow Framework (IFF) is being designed to provide a framework to address these issues.
- A preliminary description of the IFF was presented at CT04 in Vancouver. The CT06 presentation will discuss a new, modular, more mature architecture.

[1] The main IFF webpage is located at `http://suo.ieee.org/IFF/`.

1.1 Roles

pure develop category theory (the central role, this conference, for example)	**applied** apply category theory (mathematics, programming languages, concurrency, knowledge engineering, . . .)
philosophical explain/justify category theory (historical & social forces, its position in reality, . . .)	**support** implement category theory (ontologies, logical code, grammars, programming code, . . .)

- All category theory activities involve several of these roles.
- At any particular time, an activity may emphasize a certain role.
- IFF development: applied ⇒ support
  - applied: apply category theory to knowledge engineering
  - support: apply knowledge engineering to category theory

1.2 The Information Flow Framework (IFF)

- The IFF originated from a desire to define morphisms for concept lattices, based upon the isomorphism between FCA concept lattices and IF classifications.

- The IFF was develop under the auspices of the IEEE Standard Upper Ontology (SUO) project — the idea was to used category theory for representation and integration. It was the first of several approved SUO resolutions.

- There was always a close connection between the goals of the IFF and the theory of institutions.

- There was also a connection to foundations, since from the category-theoretic perspective, a strong requirement of the IFF formalism was the complete incorporation of various structures in large (level 2) categories $\mathcal{C}$, such as the pullback square

$$\begin{array}{ccc} \mathsf{mor}(\mathcal{C}) \times_{\mathsf{obj}(\mathcal{C})} \mathsf{mor}(\mathcal{C}) & \xrightarrow{\pi_1^{\mathcal{C}}} & \mathsf{mor}(\mathcal{C}) \\ {\scriptstyle \pi_0^{\mathcal{C}}} \downarrow & & \downarrow {\scriptstyle \partial_0^{\mathcal{C}}} \\ \mathsf{mor}(\mathcal{C}) & \xrightarrow[\partial_1^{\mathcal{C}}]{} & \mathsf{obj}(\mathcal{C}) \end{array}$$

  which defines the source of the composition map.

- The IFF follows two design principles

  - **conceptual warrant**
    the IFF is designed bottom-up as an experiment in foundations.
  - **categorical design**
    the IFF develops by initially defining and constraining concepts with a first order expression and then transforming (morphing) the axiomatics to an atomic expression by introducing and axiomatizing new, more central or higher, terminology.

1.3 What is an Ontology?

Etymology: (first coined in the 17th century) from the Greek, *οντος*: *of being* (*ον*: present participle of *ειμαι*: *to be*) and -*λογια*: *science*, *study*, *theory*

Aristotle: "the science of being qua (in the capacity of) being"; hence, ontology is the science of being inasmuch as it is being, or the study of beings insofar as they exist.

Mer-Web: **1** : a branch of metaphysics concerned with the nature and relations of being **2** : a particular theory about the nature of being or the kinds of existents.

Encyclo. Brit.: the theory or study of being as such; i.e., of the basic characteristics of all reality. Ontology is synonymous with metaphysics or "first philosophy" as defined by Aristotle in the 4th century BC.

Know. Engr.: "An ontology[2] is a formal, explicit specification of a shared conceptualization. It is *an abstract model of some phenomena in the world*[3], explicitly represented as *concepts, relationships and constraints*[4], which is *machine-readable*[5] and incorporates *the consensual knowledge of some community*[6]."

- The Gene Ontology
  - **concepts:** (technical: gene, protein, metabolic pathway, ...) (organization: people, papers, conferences, ...)
  - **relations:** regulator (gene) predicate, ...
- A Category Theory Ontology
  - **concepts:** category, adjunction, ...
  - **relations:** small complete predicate, object set map, subcategory relation, ...

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[2]Ontologies can be thought of as taxonomies, logical theories or knowledge-bases.
[3]semantic conceptualization
[4]logic-oriented
[5]formal and explicit
[6]shared and relative

Concepts = Types = Entities

- highway = road
- geographical feature
  - location = point
    - exit
    - interchange
    - town
    - rest-area
  - line = linear feature
    - creek
    - river
    - railroad
  - area
    - lake
    - mountain
    - city
    - county
    - state = province
    - country
- territorial division
  - county
  - state
  - country
- urban area
  - town
  - city

Predicates = Parts

principal	:	highway
toll-road	:	highway
freeway	:	highway
scenic	:	highway
is-capital	:	urban-area

Functions = Maps

name(number)	:	highway $\rightarrow$ name-tag$\times$number
number-of-lanes	:	highway $\rightarrow$ number
distance	:	point$\times$point $\rightarrow$ number
facility	:	rest-area $\rightarrow$ facility-tag
intersection	:	$\mathtt{ext}$(crosses) $\rightarrow$ point
exit-location	:	exit $\rightarrow$ highway$\times$number
lies-in	:	county $\rightarrow$ state

name-tag = $\{\mathtt{interstate}, \mathtt{state}, \mathtt{county}\}$
facility-tag = $\{\mathtt{full}, \mathtt{partial}, \mathtt{none}\}$

Relations

crosses	:	line $\rightharpoondown$ line
traverses	:	highway $\rightharpoondown$ territorial-division
goes-through	:	road $\rightharpoondown$ urban-area

Axioms

$\forall_{x,y}$ (($x, y \in$ linear feature)
crosses(x, y) $\Rightarrow$ crosses(y, x))
$\forall_{h,c,s}$ (($h \in$ highway, $c \in$ county, $s \in$ state)
(traverses(h, c) & lies-in(c, s)) $\Rightarrow$ traverses(h, s))
$\forall_{x,y,z}$ (($x, y, z \in$ location)
distance(x, z) $\leq$ distance(x, y) + distance(y, z))

Figure 1: An Ontology of Roadmaps

1.4 Design Principles

During the IFF development, two design principles have emerged as important.

- **Conceptual Warrant:** [content][7]
  IFF terminology requires conceptual warrant. Warrant means evidence for or token of authorization. Conceptual warrant is an adaptation of the librarianship notion of literary warrant.

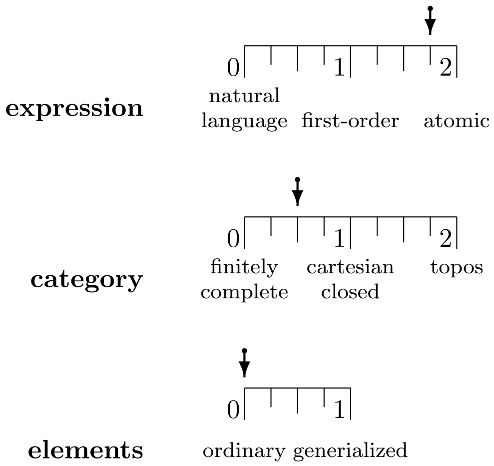
  - According to the Library of Congress (LOC), its collections serve as the literary warrant (i.e., the literature on which the controlled vocabulary is based) for the LOC subject headings system.
  - Likewise for the IFF, any term should reference a concept needed in a lower (metalevel) or more peripheral axiomatization.

LOC	subject headings	collections
IFF	higher terms	lower concepts

- **Categorical Design:** [form]
  IFF module design should follow good category-theoretic intuitions.

  - *Axiomatizations should complete any implicit ideas.* For example, any implicit adjunctions should be formalized explicitly.
    - Any current axiomatization may only be partially completed.
  - *Axiomatizations should be atomic.* Thus, axiomatizations should be in the form of declarations, equations or relational expressions. No axioms should use explicit logical notation: no variables, quantifications or logical connectives should be used.
    - Steps in the IFF axiomatization process:
      natural language ⇒ first order ⇒ atomic.
    - Although the metashell axiomatization uses first order expression, the natural part axiomatization is (destined to be) atomic.

[7] A non-starter during the IFF development was a topos axiomatization. This received objections from the SUO working group, in part due to its lack of support by motivating examples. Rejection of the topos axiomatization prompted the idea of conceptual warrant.

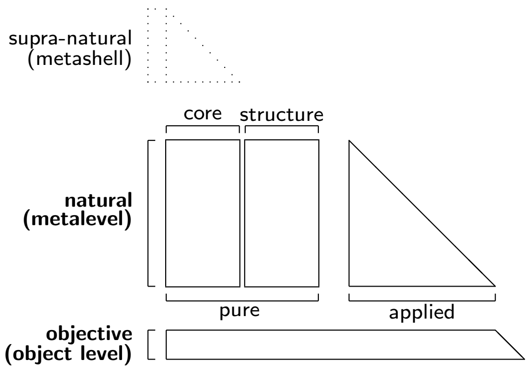


- Axioms in the metashell are in first order form; most axioms in the natural part are in atomic form.
- Finite limits have been axiomatized and applied; exponents have been axiomatized and subobjects have been partialy axiomatized, but neither has yet been applied.
- No generalized elements (morphisms) have been explicitly used in place of ordinary (global) elements; some generalized elements show up as parameters.

Figure 2: Current Development State (June 2006)

1.5 Development State

- As Heraclitus said "Everything flows, nothing stands still." So too, the IFF development is constantly under revision.
- Attention and activity has moved from applications of institution theory to a category theory standard.
- Several concepts about development have emerged
  - design principles
  - architectural framework
  - concurrent development processes
    (Figure 2 indicates degree of completion)
    - axiomatic expression: natural language ⇒ first order ⇒ atomic
      ☞ **Design Principles** page (transparency)
    - finitely complete category ⇒ cartesian-closed category ⇒ topos
      ☞ **Pure Aspect** and **Topos** pages
    - ordinary elements ⇒ generalized elements (morphisms)
      ☞ **Inclusion/Membership** and **Analogs** pages



(iconic version)

Figure 3: The IFF Architecture

2 Architecture

- a two dimensional structure consisting of levels (vertical dimension), namespaces (horizontal dimension) and meta-ontologies (composites)
- described in terms of parts, aspects and components

2.1 Modular Structure

parts: (vertical dimension)

objective part $(n = 0)$ (atomic expression; no logical structure)
terminology for object-level ontologies

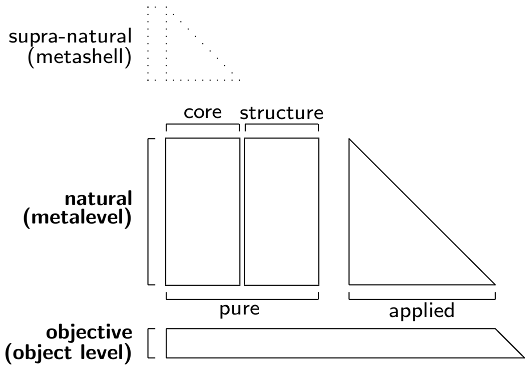
natural part $(1 \leq n < \infty)$ (atomic expression)
namespaces for many concepts of mathematics and logic

supranatural part $(n \in \{\mathsf{meta}, \mathsf{type}, \mathsf{kind}, \mathsf{iff}\})$ (first order expression)
metashell axiomatization
(temporary scaffolding for construction of the architecture)

aspects: (horizontal dimension)

pure aspect set-theoretic and category-theoretic foundations

applied aspect terminology and axiomatization for logical and semiotic functionality



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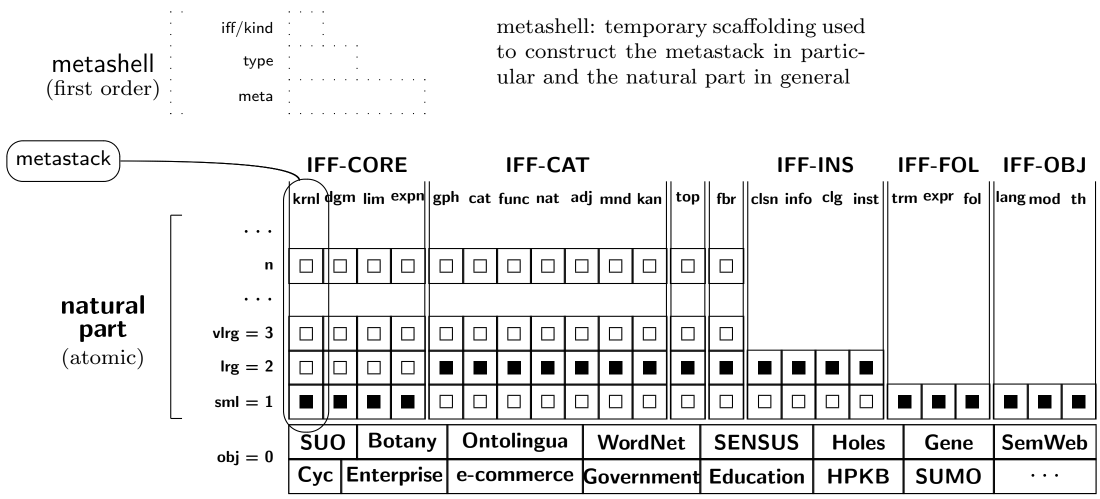
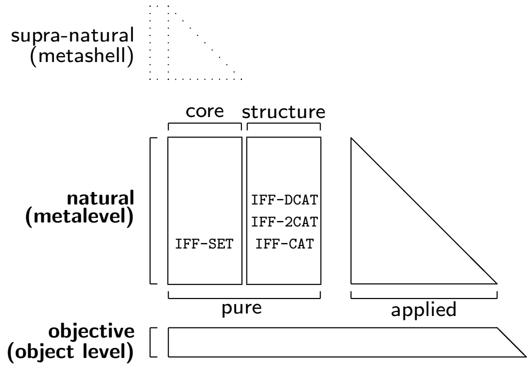


(detailed version)

Figure 4: The IFF Architecture

$$
\begin{array}{rclrcll}
 & & & \mathsf{Cat}_1 & = & \mathsf{cat}(\mathsf{Set}_1) & \text{small categories} \\
\mathsf{Set}_1 & \in & \mathsf{obj}(\mathsf{Cat}_2) & \mathsf{Cat}_2 & = & \mathsf{cat}(\mathsf{Set}_2) & \text{large categories} \\
 & & & & & \cdots & \\
\mathsf{Set}_{n-1} & \in & \mathsf{obj}(\mathsf{Cat}_n) & \mathsf{Cat}_n & = & \mathsf{cat}(\mathsf{Set}_n) & \text{level } n \text{ categories} \\
\mathsf{Set}_n & \in & \mathsf{obj}(\mathsf{Cat}_{n+1}) & \mathsf{Cat}_{n+1} & = & \mathsf{cat}(\mathsf{Set}_{n+1}) & \text{level } n{+}1 \text{ categories} \\
 & & & & & \cdots &
\end{array}
$$



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(The IFF Architecture)

2.2 Pure Aspect

"Such is 'set theory' in the practice of mathematics;
it is part of the essence from which organization emerges."
∼ Bill Lawvere

- partitioned into a core component and a structural component
- the axiomatization for any concept is given in one generic module (namespace) at level $1 \leq n < \infty$
- the finite metalevels, $1 \leq n < \infty$, are populated by generic[8] and parametric[9] meta-ontologies
- only one copy of a meta-ontology with a level parameter is needed for all finite levels

core component

contains a single generic meta-ontology `IFF-SET` for set theory, which incorporates the specialization of the meta namespace (`IFF-META`) from the metashell. The `IFF-SET` specifies set theory as a chain of toposes of Cantorian featureless abstract sets[10]

$$\mathsf{Set} = \langle \mathsf{Set}_1 \subset \mathsf{Set}_2 \subset \cdots \subset \mathsf{Set}_n \subset \cdots \rangle$$

where Set_1 contains "small" sets and functions between "small" sets.

structure component

contains various generic meta-ontologies for category theory, (`IFF-CAT`, `IFF-2CAT`, `IFF-DCAT`, ...). The `IFF-CAT` meta-ontology specifies category theory as a chain of internal categories[11]

$$\mathsf{Cat} = \langle \mathsf{Cat}_1 \subset \mathsf{Cat}_2 \subset \cdots \subset \mathsf{Cat}_n \subset \cdots \rangle$$

in the toposes Set

---

[8]generic: the terminology and axiomatization for any two metalevels is identical
[9]parametric: the metalevel index is a parameter
[10]motivated by and compatible with the *Cantorian Expansion* of sets
[11]by axiomatizing categories, functors, natural transformations, adjunctions, monads, ...

IFF Term	Concept
`#n:Set`	Set_n

`#n.set:set`
`#n.set:power`
`#n.set:{zero, one, two}`
	$\mathsf{obj}(\mathsf{Set}_n)$
	$\wp_{\mathsf{Set}_n}$
	$0_{\mathsf{Set}_n}, 1_{\mathsf{Set}_n}, 2_{\mathsf{Set}_n} = \Omega_{\mathsf{Set}_n}$

`#n.ftn:function`
`#n.ftn:power`
`#n.ftn:composition`
`#n.ftn:identity`
	$\mathsf{mor}(\mathsf{Set}_n)$
	$\wp_{\mathsf{Set}_n}$
	$\circ_{\mathsf{Set}_n}$
	1_{Set_n}

`#n.pred:fiber` `#n.pred:binary-meet`	$\mathsf{Sub}_{\mathsf{Set}_n}$ $\wedge_{\mathsf{Set}_n}$
`#n.rel:fiber01`	$\varphi_{01}^{\mathsf{Set}_n}$
`#n.lim.prd2.obj:product` `#n.lim.prd2.obj:projection{0,1}`	$\times_{\mathsf{Set}_n}$ $\pi_i^{\mathsf{Set}_n}, \quad i = 0, 1$

`#n.exp.obj:exponent`
`#n.exp.obj:evaluation`
`#n.exp.obj:hom`
`#n.exp.obj:curry`
	B^A
	$B^A \times A \to B$
	$\mathsf{Set}_n[\text{-},\text{-}]$
	$\mathsf{Set}_n[C \times A, B] \to \mathsf{Set}_n[C, B^A]$

Table 1: Set_n as a Topos

2.3 Topos Elements

- A category $\mathcal{E}$ is a *topos* when it has finite limits, it is cartesian closed, and it has a subobject classifier; equivalently, when it has finite limits and comes equipped with

  - an object of truth values $\Omega_{\mathcal{E}} \in \mathsf{obj}(\mathcal{E})$,
  - a power function $\wp_{\mathcal{E}} : \mathsf{obj}(\mathcal{E}) \to \mathsf{obj}(\mathcal{E})$,
  - for each object $A \in \mathcal{E}$ two natural isomorphisms

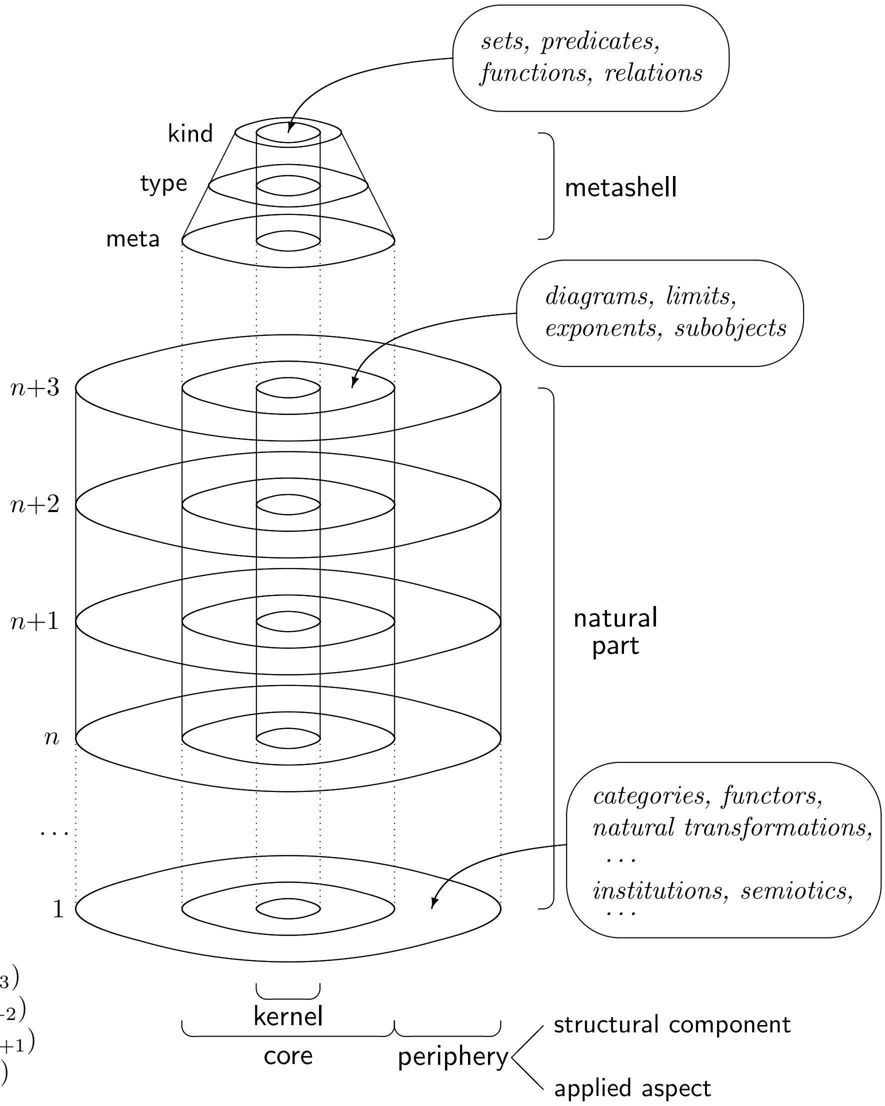
$$\begin{aligned} \mathsf{Sub}_{\mathcal{E}}(A) &\cong \mathsf{Hom}_{\mathcal{E}}[A, \Omega_{\mathcal{E}}] \\ \mathsf{Hom}_{\mathcal{E}}[A \times B, \Omega_{\mathcal{E}}] &\xrightarrow{\varphi_{01}^{\mathcal{E}}} \mathsf{Hom}_{\mathcal{E}}[A, \wp_{\mathcal{E}} B] \end{aligned}$$

  where $\mathsf{Sub}_{\mathcal{E}} A$ is the set of subobjects of A.

Fact 1 *For each n, the category Set_n of level n sets and functions is a topos.*

Proof: See Table 1, which contains selected topos-representing terms. ∎

2.4 Graphic



$$
\begin{aligned}
|\texttt{(\#n+2).rel:restriction}| &\in \mathsf{rel}(\mathsf{Set}_{n+3}) \\
|\texttt{(\#n+1).set:set}| &\in \mathsf{obj}(\mathsf{Set}_{n+2}) \\
|\texttt{\#n.ftn:source}| &\in \mathsf{mor}(\mathsf{Set}_{n+1}) \\
|\texttt{f}| &\in \mathsf{mor}(\mathsf{Set}_{n})
\end{aligned}
$$

```
((#n+1).set:set #n.set:set)
((#n+2):subset #n.set:set (#n+1).set:set)
(not (#n.set:set #n.set:set))
```

```
((#n+1).set:set #n.ftn:function)
((#n+2):subset #n.ftn:function (#n+1).ftn:function)

((#n+1).ftn:function #n.ftn:source)
(= ((#n+1).ftn:source #n.ftn:source) #n.ftn:function)
(= ((#n+1).ftn:target #n.ftn:source) #n.set:set)
((#n+2):restriction #n.ftn:source (#n+1).ftn:source)

((#n+1).ftn:function #n.ftn:target)
(= ((#n+1).ftn:source #n.ftn:target) #n.ftn:function)
(= ((#n+1).ftn:target #n.ftn:target) #n.set:set)
((#n+2):restriction #n.ftn:target (#n+1).ftn:target)
```

```
(#n.set:set A)
(#n.set:set B)
(#n.ftn:function f)
(= (#n.ftn:source f) A)
(= (#n.ftn:target f) B)
```

$$\begin{array}{ccccccccccccc}
\cdots & \subseteq & \mathsf{set}_n & \subseteq & \mathsf{set}_{n+1} & \subseteq & \cdots & \subseteq & \mathsf{set}_{\mathsf{meta}} & \subseteq & \mathsf{set}_{\mathsf{type}} & \subseteq & \mathsf{set}_{\mathsf{iff}} \\
\cdots & \subseteq & \mathsf{ftn}_n & \subseteq & \mathsf{ftn}_{n+1} & \subseteq & \cdots & \subseteq & \mathsf{ftn}_{\mathsf{meta}} & \subseteq & \mathsf{ftn}_{\mathsf{type}} & \subseteq & \mathsf{ftn}_{\mathsf{iff}} \\
\cdots & \subseteq & \mathsf{pred}_n & \subseteq & \mathsf{pred}_{n+1} & \subseteq & \cdots & \subseteq & \mathsf{pred}_{\mathsf{meta}} & \subseteq & \mathsf{pred}_{\mathsf{type}} & \subseteq & \mathsf{pred}_{\mathsf{iff}} \\
\cdots & \subseteq & \mathsf{rel}_n & \subseteq & \mathsf{rel}_{n+1} & \subseteq & \cdots & \subseteq & \mathsf{rel}_{\mathsf{meta}} & \subseteq & \mathsf{rel}_{\mathsf{type}} & \subseteq & \mathsf{rel}_{\mathsf{iff}} \\
\cdots & \subseteq & \mathsf{endo}_n & \subseteq & \mathsf{endo}_{n+1} & \subseteq & \cdots & \subseteq & \mathsf{endo}_{\mathsf{meta}} & \subseteq & \mathsf{endo}_{\mathsf{type}} & \subseteq & \mathsf{endo}_{\mathsf{iff}}
\end{array}$$

Subset

function		
∂_i	:	source/target
(∂_i)	:	their pairing
predicate		
γ	:	genus
δ	:	differentia
relation		
σ_i	:	component sets
(σ_i)	:	their pairing
ε	:	extent

$$\begin{array}{ccccccccccccc}
\cdots & \sqsubseteq & \partial_0^n & \sqsubseteq & \partial_0^{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \partial_0^{\mathsf{meta}} & \sqsubseteq & \partial_0^{\mathsf{type}} & \sqsubseteq & \partial_0^{\mathsf{iff}} \\
\cdots & \sqsubseteq & \partial_1^n & \sqsubseteq & \partial_1^{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \partial_1^{\mathsf{meta}} & \sqsubseteq & \partial_1^{\mathsf{type}} & \sqsubseteq & \partial_1^{\mathsf{iff}} \\
\cdots & \sqsubseteq & \delta_n & \sqsubseteq & \delta_{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \delta_{\mathsf{meta}} & \sqsubseteq & \delta_{\mathsf{type}} & \sqsubseteq & \delta_{\mathsf{iff}} \\
\cdots & \sqsubseteq & \varepsilon_n & \sqsubseteq & \varepsilon_{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \varepsilon_{\mathsf{meta}} & \sqsubseteq & \varepsilon_{\mathsf{type}} & \sqsubseteq & \varepsilon_{\mathsf{iff}} \\
\cdots & \sqsubseteq & \sigma_0^n & \sqsubseteq & \sigma_0^{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \sigma_0^{\mathsf{meta}} & \sqsubseteq & \sigma_0^{\mathsf{type}} & \sqsubseteq & \sigma_0^{\mathsf{iff}} \\
\cdots & \sqsubseteq & \sigma_1^n & \sqsubseteq & \sigma_1^{n+1} & \sqsubseteq & \cdots & \sqsubseteq & \sigma_1^{\mathsf{meta}} & \sqsubseteq & \sigma_1^{\mathsf{type}} & \sqsubseteq & \sigma_1^{\mathsf{iff}}
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \dot\sqsubseteq & (\partial^n_{i=0,1}) & \dot\sqsubseteq & (\partial^{n+1}_{i=0,1}) & \dot\sqsubseteq & \cdots & \dot\sqsubseteq & (\partial^{\mathsf{meta}}_{i=0,1}) & \dot\sqsubseteq & (\partial^{\mathsf{type}}_{i=0,1}) & \dot\sqsubseteq & (\partial^{\mathsf{iff}}_{i=0,1}) \\
\cdots & \dot\sqsubseteq & \gamma_n & \dot\sqsubseteq & \gamma_{n+1} & \dot\sqsubseteq & \cdots & \dot\sqsubseteq & \gamma_{\mathsf{meta}} & \dot\sqsubseteq & \gamma_{\mathsf{type}} & \dot\sqsubseteq & \gamma_{\mathsf{iff}} \\
\cdots & \dot\sqsubseteq & (\sigma^n_{i=0,1}) & \dot\sqsubseteq & (\sigma^{n+1}_{i=0,1}) & \dot\sqsubseteq & \cdots & \dot\sqsubseteq & (\sigma^{\mathsf{meta}}_{i=0,1}) & \dot\sqsubseteq & (\sigma^{\mathsf{type}}_{i=0,1}) & \dot\sqsubseteq & (\sigma^{\mathsf{iff}}_{i=0,1})
\end{array}$$

(Optimal-)Restriction

$\subseteq$	:	subset	
$\leq$	:	delimitation	(pred)
$\sqsubseteq$	:	restriction	(ftn)
$\preceq$	:	abridgment	(rel)

$$\begin{array}{ccccccccccccc}
\cdots & \preceq & \subseteq_n & \preceq & \subseteq_{n+1} & \preceq & \cdots & \preceq & \subseteq_{\mathsf{meta}} & \preceq & \subseteq_{\mathsf{type}} & \preceq & \subseteq_{\mathsf{iff}} \\
\cdots & \preceq & \leq_n & \preceq & \leq_{n+1} & \preceq & \cdots & \preceq & \leq_{\mathsf{meta}} & \preceq & \leq_{\mathsf{type}} & \preceq & \leq_{\mathsf{iff}} \\
\cdots & \preceq & \sqsubseteq_n & \preceq & \sqsubseteq_{n+1} & \preceq & \cdots & \preceq & \sqsubseteq_{\mathsf{meta}} & \preceq & \sqsubseteq_{\mathsf{type}} & \preceq & \sqsubseteq_{\mathsf{iff}} \\
\cdots & \preceq & \preceq_n & \preceq & \preceq_{n+1} & \preceq & \cdots & \preceq & \preceq_{\mathsf{meta}} & \preceq & \preceq_{\mathsf{type}} & \preceq & \preceq_{\mathsf{iff}}
\end{array}$$

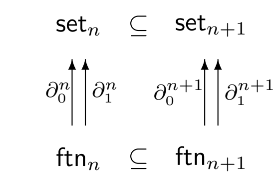
Abridgment

Table 2: Kernel Chains

just as (binary) relations are predicates (unary relations or parts) on a binary product and predicates are special functions (injections), so also abridgment is a special case of delimitation and delimitaton is a special case of optimal-restriction

2.5 Metastack

- The metastack is the kernel of the core component
  - ➤ represents the Cantorian Expansion
  - ➤ lattice-like structure (Figure 2): subset, restriction, delimitation and abridgment chains
  - ➤ binds/anchors natural part, connects natural part to metashell



$\mathsf{set}_n, \mathsf{ftn}_n \in \mathsf{set}_{n+1}$
$\partial_0^n, \partial_1^n, \circ^n, 1^n \in \mathsf{ftn}_{n+1}$

	Math	IFF	
names			
	a	`o.i:a`	inner context i outer context o
atoms and terms			
set element	$x \sqsubseteq X$ or $X(x)$	`(X x)`	
predicate member	$x \in b$ or $b(x)$	`(b x)`	
relation member	$(x, y) \in r$ or $r(x, y)$	`(r x y)`	
function application	$f(x)$	`(f x)`	
equations			
	$\sigma = \tau$ or $=(\sigma, \tau)$	`(= s t)`	
connectives			
and	$\phi \wedge \psi$ or $\wedge(\phi, \psi)$	`(and P Q)`	
not	$\neg\phi$ or $\neg(\phi)$	`(not P)`	
quantifiers			
universal	$\forall_{x_0 \in X_0, x_1 \in X_1} \phi$		
	or $\forall_{(x_0 \in X_0, x_1 \in X_1)}(\phi)$	`(forall (x0 (X0 x0) x1 (X1 x1)) P)`	

Table 3: Syntax Tutorial

2.6 IFF Syntax

- The LISt Processing (LISP) programming language is the second oldest (1958). All program code is written as parenthesized lists.
- The Knowledge Interchange Format (KIF), which has a LISP-like format, was created to serve as a syntax for first-order logic.
- The IFF logical notation, which is a vastly simplified and modified version of KIF, also has a LISP-like format.
- The IFF grammar is located at `http://suo.ieee.org/IFF/grammar.pdf`.
  - Written in Extended Backus Naur Form (EBNF), a convenient way to describe the grammar of a language.
  - Features of the IFF language:
    - contains both logical IFF code and comments
    - nested namespace assumptions
    - levels specified by prefixes

3 Foundations

"All of the substance of mathematics can be fully expressed in categories."
∼ Bill Lawvere

3.1 Two Misconceptions

Myth: "Category theory is the 'insubstantial part' of mathematics and it heralds an era when precise axioms are no longer needed."[12]

Myth: "There are 'size problems' if one tries to do category theory in a way harmonious with the standard practice of professional set theorists."

▼ "The first of these misunderstandings is connected with taking seriously the jest 'sets without elements'. The traditions of algebraic geometry and of category theory are completely compatible about elements."

▲ The following transparencies address this issue.

☞ **Inclusion and Membership**

☞ **Analogs**

▼ "Contrary to Fregean rigidity, in mathematics we never use 'properties' that are defined on the universe of 'everything'. There is the 'universe of discourse' principle which is very important: for example, any given group, (or any given topological space, etc.) acts as a universe of discourse."

▲ The IFF syntax addresses this issue. It requires the use of *restricted quantification* in logical expression. For example, the following IFF code axiomatizes the inverse element for a group:

```
(forall (?G (group ?G))
    (forall (?a (?G ?a))
        (and (= ((multiplication ?G) [?a ((inverse ?G) ?a)]) (unit ?G))
             (= ((multiplication ?G) [((inverse ?G) ?a) ?a]) (unit ?G)))))
```

▲ The following transparencies address the second misunderstanding.

☞ **Cantor**

☞ **Unions and Universes**

[12]Taken from Bill Lawvere's 3 messages to the CAT list: *Why are we concerned?*

3.2 Cantor

From the book *Sets for Mathematics* by Bill Lawvere and Robert Rosebrugh.

Definition 1 *Let Y be any set. An element $y \in Y$ is a* fixed point *of an endofunction $\tau : Y \to Y$ when $\tau(y) = y$. A set Y has the* fixed point property *when every endofunction on Y has at least one fixed point.*

Theorem 1 *Suppose there is a set X and a function $\varphi : X \times X \to Y$ whose curry $\hat{\varphi} : X \to Y^X$, where $\hat{\varphi}(a) = \varphi(a, \text{-})$ for all $a \in X$, is surjective; that is, such that for every function $f : X \to Y$ there is at least one element $a \in X$ such that $f = \hat{\varphi}(a) = \varphi(a, \text{-})$. Then Y has the fixed point property.*

Corollary 1 (Cantor) *If a set Y has at least one endofunction $\tau : Y \to Y$ with no fixed points, then for every set X there is no surjection $X \to Y^X$.*

Corollary 2 *For any set X,*

$$X < 2^X = \wp X.$$

Corollary 3 *There cannot exist a "universal set" U for which every set X is a subset $X \subseteq U$.*

Proof: If so, then the inclusion $X \to U$ is an injection. Hence, the exponent map $2^U \to 2^X$ is a surjection. Define $X = 2^U$ to get a contradiction. ■

Corollary 4 *The collection* set *of all sets is not a set.*

Proof: If set were a set, then $U = \bigcup \mathsf{set}$ would be a "universal set". ■

Comment The sets here are called "small" sets. The collection of small sets, like the set of natural numbers $\aleph$, is either defined naturally, by convention or logically/mathematically[13]. This corollary states that there are sets that are not small. Change the notation, letting set_1 denote the collection of small sets, and set_2 denote the collection of sets either small or not (call them "large" sets). So that $\mathsf{set}_1 \subseteq \mathsf{set}_2$, $\mathsf{set}_1 \in \mathsf{set}_2$, but $\mathsf{set}_1 \notin \mathsf{set}_1$.

Corollary 5 (Cantorian Expansion) *The collection of all sets unfolds into a chain (of Cantorian featureless abstract sets)*

$$\mathsf{set} = \langle \mathsf{set}_1 \subset \mathsf{set}_2 \subset \cdots \subset \mathsf{set}_n \subset \cdots \subset \mathsf{set}_\infty \rangle$$

where set_1 denotes the collection of all "small" sets.

Proof: Starting from the small sets set_1, apply Corollary 4 repeatedly. ■

[13] The set of natural numbers, which occurs in nature, was used in antiquity (convention) and axiomatized in modern times (logic/math).

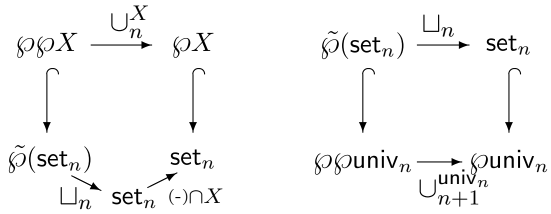


Figure 5: Local and Bounded Unions

3.3 Unions and Universes

- Let n be any metalevel and let set_n be the collection of all level n sets[14].
  - For any level n set $X \in \mathsf{set}_n$, the *bounded union* operation[15]

    $$\bigcup_n^X : \wp\wp X \to \wp X$$

    is defined as $\bigcup_n^X(\mathcal{X}) = \{x \in X \mid \exists_Y (x \in Y \in \mathcal{X})\}$ for $\mathcal{X} \in \wp\wp X$. Define $\tilde{\wp}(\mathsf{set}_n) = \mathsf{set}_n \cap \wp\mathsf{set}_n \in \mathsf{set}_{n+1}$. Then, $\wp\wp X \subseteq \tilde{\wp}(\mathsf{set}_n)$.
  - A level n *universe* is a level $n{+}1$ set $U \in \mathsf{set}_{n+1}$ that has the properties: $\mathsf{set}_n \subseteq U$ "every level n set is an element of the universe" and $\mathsf{set}_n \subseteq \wp U$ "every level n set is a subset of the universe". Then, $\tilde{\wp}(\mathsf{set}_n) \subseteq \wp\wp U$.
  - The *local union* operation[16]

    $$\bigsqcup_n : \tilde{\wp}(\mathsf{set}_n) \to \mathsf{set}_n$$

    is defined as $\bigsqcup_n(\mathcal{X}) = \{x \mid \exists_Y (x \in Y \in \mathcal{X})\}$[17] $\mathcal{X} \in \tilde{\wp}(\mathsf{set}_n)$. Letting $\mathcal{X} = \mathsf{set}_n$, define $\mathsf{univ}_n = \bigsqcup_{n+1} \mathsf{set}_n = \{x \mid \exists_Y (x \in Y \in \mathsf{set}_n)\} \in \mathsf{set}_{n+1}$. This is a specific level n universe.

local unions define bounded unions: (left side Figure 5)

bounded unions define local unions: (right side Figure 5)

---

[14]We assume that set_n is closed under subset order ($X \in \mathsf{set}_n, Y \subseteq X$ implies $Y \in \mathsf{set}_n$), any level n set is a level $n{+}1$ set ($\mathsf{set}_n \subseteq \mathsf{set}_{n+1}$), set_n is itself a level $n{+}1$ set ($\mathsf{set}_n \in \mathsf{set}_{n+1}$) and set_n is not a level n set ($\mathsf{set}_n \notin \mathsf{set}_n$).

[15]In the current version of the IFF, although we have bounded unions at every metalevel, except for Cantor we have not used either local (unbounded) unions or universes.

[16]In the IFF, this union would be specified within and "local" to a particular IFF metalevel.

[17]Following Mac Lane in the foundations section of *CWM*([2]).

	Grothendieck universe	IFF
Axioms:	If $X \in \mathcal{U}$ and $x \in X$, then $x \in \mathcal{U}$. If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$. If $X \in \mathcal{U}$, then $\wp(X) \in \mathcal{U}$. If $I \in \mathcal{U}$ and $X_\alpha \in \mathcal{U}$ for each $\alpha \in I$, then $\bigcup_{\alpha \in I} X_\alpha \in \mathcal{U}$.	If $X \in \mathsf{set}_n$ and $x \in X$, then $x \in \mathsf{univ}_n$. doubleton function $\{\text{-},\text{-}\}_X : X^2 \to \wp(X)$ power set function $\wp : \mathsf{set}_n \to \mathsf{set}_n$ bounded union $\bigcup_n^X : \wp\wp X \to \wp X$, and local union $\coprod_n : \tilde{\wp}(\mathsf{set}_n) \to \mathsf{set}_n$
Theorems:	If $X \in \mathcal{U}$ and $Y \subseteq X$, then $Y \in \mathcal{U}$. If $X, Y \in \mathcal{U}$, then $(f : X \to Y) \in \mathcal{U}$. If $X \in \mathcal{U}$ and $Y \cong X$, then $Y \in \mathcal{U}$. If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$. If $I \in \mathcal{U}$ and $X_\alpha \in \mathcal{U}$ for each $\alpha \in I$, then $\coprod_{\alpha \in I} X_\alpha, \prod_{\alpha \in I} X_\alpha, \bigcap_{\alpha \in I} X_\alpha \in \mathcal{U}$.	If $X \in \mathsf{set}_n$ and $Y \subseteq X$, then $Y \in \mathsf{set}_n$. If $X, Y \in \mathsf{set}_n$, then $(f : X \to Y) \in \mathsf{ftn}_n$. If $X \in \mathsf{set}_n$ and $Y \cong X$, then $Y \in \mathsf{set}_n$. singleton function $\{\text{-}\}_X : X \to \wp(X)$ The category set_n is small (co)complete. The preorder $\wp(X)$ is a Boolean algebra.

Table 4: Grothendieck-IFF Analogs

Grothendieck Universes. The IFF has much in common with Grothendieck universes. A Grothendieck universe $\mathcal{U}$ is meant to provide a set in which all of mathematics can be performed[18]. The IFF provides a framework in which all of mathematics can be axiomatized. Grothendieck universes model universes of sets. However, IFF universes contain non-set objects such as functions, predicates, relations, vectors, numbers, ships, stars, pelicans and bacteria. This means that Grothendieck universes are more like the toposes $\langle \mathsf{Set}_n, 1 \leq n < \infty \rangle$ than the IFF universes $\langle \mathsf{univ}_n, 1 \leq n < \infty \rangle$. Indeed, the main intuition is that for any set X, there is a Grothendieck universe $\mathcal{U}$ with $X \in \mathcal{U}$. Similarly, for any IFF set X, there is a whole number $1 \leq n < \infty$ with $X \in \mathsf{set}_n = \mathsf{obj}(\mathsf{Set}_n)$. More precisely, a *Grothendieck universe* $\mathcal{U}$ is a set which is closed under membership, and contains doubletons, powers and indexed unions. These axioms imply that a Grothendieck universe $\mathcal{U}$ is closed under the subset order, and contains functions, isomorphs, singletons, indexed coproducts (disjoint unions), indexed products and indexed intersections. Analogs between Grothendieck universes and the IFF are listed in Table 4.

[18](by way of Wikipedia) Bourbaki, N., *Univers*, appendix to Expos I of Artin, M., Grothendieck, A., Verdier, J. L., eds., *Théorie des Topos et Cohomologie Étale des Schémas (SGA 4)*, second edition, Springer-Verlag, Heidelberg, 1972.

element	$x \sqsubseteq X$
part	$b : X$

belongs	$x \sqsubseteq y$
inclusion	$a \subseteq b$
member	$x \in b$

3.4 Inclusion/Membership

■ Let $\mathcal{C}$ be any *category*[19] with $X \in \mathsf{obj}(\mathcal{C})$ be any $\mathcal{C}$-object. For any morphism $x \in \mathsf{mor}(\mathcal{C})$, x is an *element* of X, $x \sqsubseteq X$, when $X = \partial_1^{\mathcal{C}}(x)$.

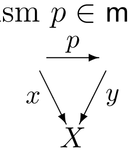
$$\xrightarrow{x} X$$

Let $\Xi_{\mathcal{C}}(X) = \mathsf{obj}(\mathcal{C}{\downarrow}X)$ denote the set of elements of X. For any X-element $b \in \Xi_{\mathcal{C}}(X)$, b is a *part* of X[20], $b : X$, when b is a monomorphism.

$$\overset{b}{\hookrightarrow} X$$

Let $\wp_{\mathcal{C}}(X) = \mathsf{pred}_{\mathcal{C}}(X) \subseteq \Xi_{\mathcal{C}}(X)$ denote the set of parts of X.

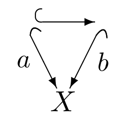
- For any two X-elements $x, y \in \Xi_{\mathcal{C}}(X)$, x *belongs* to y, $x \sqsubseteq y$, when there exists a *proof* morphism $p \in \mathsf{mor}(\mathcal{C})$ such that $x = p \cdot y$[21].



slice category
$\mathcal{C}{\downarrow}X$

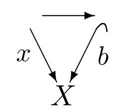
When y is an X-part, the proof p of that belonging is unique.

- For any two X-parts $a, b \in \wp_{\mathcal{C}}(X)$, a *is included in* b, $a \subseteq b$, when a belongs to b.



subobject preorder
$\wp_{\mathcal{C}}(X) = \langle \wp_{\mathcal{C}}(X), \leq_X^{\mathcal{C}} \rangle$

- For any X-element $x \in \Xi_{\mathcal{C}}(X)$ and any X-part $b \in \mathsf{pred}_{\mathcal{C}}(X)$, x is a *member* of b, $x \in b$, when x belongs to b.



distributor
$\mathcal{C}{\downarrow}X \overset{\in_X^{\mathcal{C}}}{\rightharpoondown} \wp_{\mathcal{C}}(X)$

Fact 2 *Inclusion equivalent to universal implication of membership*

$$a \subseteq b \ \textit{ iff } \ \forall_{x \sqsubseteq X}(x \in a \Rightarrow x \in b) \qquad \subseteq \, = \, \in \backslash \in$$

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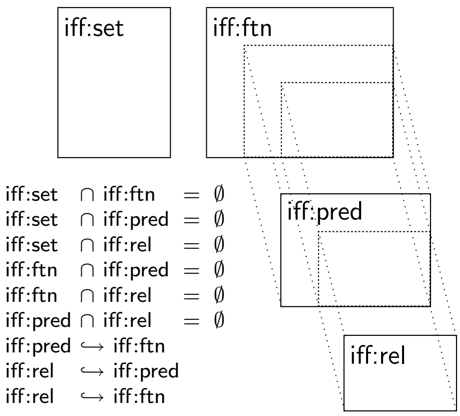
[19] The material here is adapted from Bill Lawvere's emails *Why are we concerned?*.
[20] In the IFF, b is called a *predicate* with *genus* $X = \gamma_{\mathcal{C}}(b)$ and *differentia* $\delta_{\mathcal{C}}(b)$.
[21] Here, composition is written in diagrammatic order.



Basic Collections

	Set	Function	Relation
	`thing`		
`(X x)`	`set`	`binary-intersection`	`subset`
`(f x)`	`function`	`source target`	`(optimal-)restriction`
`(b x)`	`predicate`	`genus`	`delimitation`
`(r x y)`	`relation`	`set0 set1`	`abridgment`
	`endorelation`	`base`	
	`0 1 2`		

Technical and Recommended Prefix : `iff`

Terminology

Figure 6: The IFF Namespace

3.5 The IFF Namespace

- The terminology of the IFF namespace, a tiny namespace (only 21 terms) at the very top of the IFF architecture, is listed in the right side of Figure 6.
  - IFF things fall into four basic pairwise disjoint collections (left side of Figure 6): sets, functions, predicates and relations.
  - Some of the components of these four basic collections are also introduced here: predicate genus, function source/target and relation domain/codomain.
  - The subset, delimitation, restriction, optimal-restriction and abridgment endorelations are important for building the vertical aspect of the metastack structural framework.
  - The IFF namespace is closely related to the IFF grammar. The IFF grammar can be used for type correctness in set membership, function application, and predicate/relation invocation:

Here is a fragment of code[22] from the function namespace in the kind metalevel that defines the belonging relation for the category $\mathsf{Set}_{\mathsf{kind}}$.

```
(iff.rel:endorelation belongs)
(= (iff.rel:base belongs) function)
(forall (?x (function ?x) ?y (function ?y))
    (<=> (belongs ?x ?y)
         (and (= (target ?x) (target ?y))
              (exists (?p (function ?p))
                  (= ?x (composition [?p ?y]))))))
```

[22]Unqualified names are qualified by '`kind.ftn`' via the nested namespace mechanism.

3.6 Analogs

- ordinary (global) elements (defined in grammar and iff namespace)

	notation	IFF
set	X	`(set X)`

function	$x : Z \to X$	`(function x)`
		`(= (source x) Z)`
		`(= (target x) X)`

element	$x \in X$	`(X x)`
part	$b : X$	`(predicate b)` `(= (genus b) X)`
belongs	$x \sqsubseteq y$	
inclusion	$a \subseteq b$	
member	$x \in b$ $(x, y) \in r$	`(b x)` `(r x y)`
application	$f(x)$	`(f x)`

- generalized elements (morphisms) (defined in kind namespace)

	notation	IFF
set	X	`(set X)`

function	$x : Z \to X$	`(function x)`
		`(= (source x) Z)`
		`(= (target x) X)`

element	$x \sqsubseteq X$	`(= (target x) X)`
part	$b : X$	`(part b)` `(= (genus b) X)`
belongs	$x \sqsubseteq y$	`(belongs x y)` `(belonging [x y])`
inclusion	$a \subseteq b$	`(included-in a b)` `(inclusion [a b])`
member	$x \in b$	`(member x b)` `(membership [x b])`
composition	$x \cdot f$	`(composition [x f])`

4 Concluding Remarks

4.1 Future Work

- **The IFF**

  - Finish the concurrent development processes
    - **expression**: natural language ⇒ first order ⇒ atomic
    - **category:** finitely complete ⇒ cartesian-closed ⇒ topos
    - **elements:** ordinary ⇒ generalized (morphisms)
  - Continue work on axiomatizations in the structural component.

- **The Category Theory Community**

  - All scientific communities, indeed all communities of discourse (disciplines), create their own conceptual structures with accompanying terminology and meaning (ontologies). Many communities are now working to standardize their ontologies.
  - There is also a search for a unifying framework for these endeavers. It has been suggested that category theory can serve this role — category theory can serve as a meta-ontology, an ontology of ontologies.
  - **Proposal:** The category theory community will form a working group (under the auspices of some organization or consortium) for the purpose of developing a standard ontology for category theory.

4.2 Standards

- **Why standards?**

  - Standards allow interoperability between cooperating groups in technology, business and science. Standards are intended to be documented, known descriptions of how something works so that every group who adheres to the standard can interoperate. There is no one true standard — the evolution of standards is a sign of healthy innovation.
  - The plurality of standards-issuing organizations means that some standards do not necessarily have the support of all communities[23] The standards of large communities can be created to replace the various incompatible standards of smaller communities or can be built from scratch by groups of experts.

- **What types of standards exist?**

  - **open standard** documented for all; developed and maintained by peers and in public
  - **proprietary standard** developed and maintained by/for one particular organisation (Microsoft's Windows, Adobe's PDF)
  - **ad hoc standard** more widely used than their originator intended (JVC's VHS, Compuserve's GIF)
  - **de facto standard** the property of consortia that represent a wide range of interests (IETF's HTTP protocol, W3C's HTML format, OMG's CORBA)
  - **de jure standard** developed by standards bodies established under national or international laws; (the meter of the French Academy of Sciences[24] and the Geneva Conference on Weights and Measures[25], ANSI's C programming language, ISO's JPEG)

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[23] "The nice thing about standards is that there are so many to choose from." ∼ Tanenbaum

[24] one/ten-millionth of the distance from the equator to the north pole

[25] the distance light travels in a vacuum in 1/299,792,458 seconds

Principles	**Process**

• *due process*
• *openness*
• *consensus*
• *balance*
• *right of appeal*
	0. idea
	1. project approval by standards body
	2. draft development by working group
	  a. form WG with chair and technical editor
	  b. establish goals, deadlines and schedule
	  c. draft document
	  d. reviewed by technical editor
	3. sponsor ballot
	4. standards board approval process
	5. publish standards
	6. periodically reaffirm, revise or withdraw standard; goto 1.

Table 5: IEEE Standards

Our two standard references for the IFF are the books: *Sets for Mathematics* (2003) by F. William Lawvere and Robert Rosebrugh [1] and *Categories for the Working Mathematician* (1971) by Saunders Mac Lane [2].

References

[1] F. William Lawvere and Robert Rosebrugh. *Sets for Mathematics*. Cambridge University Press, 2003.

[2] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.