THE EULER CHARACTERISTIC OF A CATEGORY

Tom Leinster

U. of Glasgow

(Partial support: Nuffield Foundation)

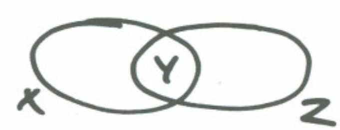
Credits: Schanuel, Rota, Baez, Dolan, ...

PLAN

0. Möbius inversion for categories

1. Answer question: given $X : \mathbb{A} \longrightarrow \mathbf{Set}$, when is $|\varinjlim X|$ determined by $(|X_a|)_{a \in \mathbb{A}}$?

   (E.g. • $|X \cup_Y Z| = |X| + |Z| - |Y|$

   • If S is a free G-set, $|S/G| = |S|/|G|$)

2. Euler characteristic of categories

0. MÖBIUS INVERSION

Definition

Let $\mathbb{A}$ be a finite category.

Defns: • $E(\mathbb{A})$ is the $\mathbb{Q}$-algebra $\{$functions $\mathrm{ob}\,\mathbb{A} \times \mathrm{ob}\,\mathbb{A} \to \mathbb{Q}\}$ with multiplication

$$(\theta \cdot \varphi)(a,c) = \sum_b \theta(a,b)\,\varphi(b,c)$$

($\theta, \varphi \in E(\mathbb{A})$, $a, c \in \mathbb{A}$), and Kronecker δ as unit.

• $\zeta \in E(\mathbb{A})$ is given by $\zeta(a,b) = |\mathbb{A}(a,b)|$

• $\mathbb{A}$ has Möbius inversion if ζ^{-1} exists; we write $\mu = \zeta^{-1}$ (the Möbius function)

So $\forall a, c$,

$$\sum_b \zeta(a,b)\mu(b,c) = \delta(a,c) = \sum_b \mu(a,b)\zeta(b,c).$$

(Compare Content–Leray–Leroux.)

Examples

- $\mathbb{A} = (1 \longrightarrow 2)$: $\quad \zeta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

- $\mathbb{A}$ is a finite monoid M: $\quad \zeta = |M|, \quad \mu = 1/|M|$

- Let $\mathbb{D}_N$ be the cat with

  objects: the finite ordinals $1, \ldots, N$

  maps: order-preserving injections.

  Then $\zeta(a,b) = \binom{b}{a}, \quad \mu(a,b) = (-1)^{b-a}\binom{b}{a}$.

- Poset $(\mathbb{Z}^+, |)$:

$$\zeta(a,b) = \begin{cases} 1 & \text{if } a|b \\ 0 & \text{otherwise}, \end{cases}$$

classical Möbius

$$\mu(a,b) = \begin{cases} \mu(b/a) & \text{if } a|b \\ 0 & \text{otherwise.} \end{cases}$$

Properties

- Any cat with Möbius inversion is skeletal ...
- ... but not every skeletal cat has M. inversion
- There are formulas for μ valid for several classes of skeletal cats:
  - posets (Rota):

$$\mu(a,b) = \sum_n (-1)^n \left| \{ \text{chains } a = a_0 < \cdots < a_n = b \} \right|$$

  - cats with no non-trivial idempotents
  - cats with an epi-mono factorization system

What is it good for?

One answer: finding the representing family of a sum of representables.

Propn: If $\mathbb{A}$ has Möbius inversion and

$$X \cong \sum_a r_a \cdot \mathbb{A}(a,-) : \quad \mathbb{A} \longrightarrow \mathbf{Set}$$

($r_a \in \mathbb{N}$) then

$$r_a = \sum_b |Xb| \, \mu(b,a).$$

Proof: $$\mathrm{RHS} = \sum_b \Big(\sum_c r_c \, \zeta(c,b) \Big) \mu(b,a)$$

$$= \sum_c r_c \Big(\sum_b \zeta(c,b) \mu(b,a) \Big)$$

$$= \sum_c r_c \, \delta(c,a)$$

$$= r_a .$$

$\square$

What is *that* good for?

One answer: counting problems.

E.g.: Let

d_n = no. of derangements of n letters

(permutations with no fixed point).

Fix $N \in \mathbb{N}$ and consider

$$\begin{aligned} S : \mathbb{D}_N &\longrightarrow \mathbf{Set} \\ n &\longmapsto S_n . \end{aligned}$$

Then

$$S_n \cong \sum_m d_m \, \mathbb{D}_N(m, n)$$

so

$$\begin{aligned} d_n &= \sum_m |S_m| \, \mu(m, n) \\ &= \sum_{0 \leq m \leq n} m! \cdot (-1)^{n-m} \binom{n}{m} \\ &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{n!} \right). \end{aligned}$$

1. THE CARDINALITY OF A COLIMIT

Recall: we're trying to generalize the following fact:

$$\text{if} \qquad \begin{tikzcd} & X_2 \\ X_1 \arrow[ur, hook] \arrow[dr, hook] & \\ & X_3 \end{tikzcd}$$

$$\text{then} \qquad |X_2 +_{X_1} X_3| = |X_2| + |X_3| - |X_1| .$$

Weightings

Let $\mathbb{A}$ be a finite category.

Defn: A weighting on $\mathbb{A}$ is a function $k^\bullet : \mathrm{ob}\,\mathbb{A} \longrightarrow \mathbb{Q}$ such that $\forall a$, $\sum_b \zeta(a,b) k^b = 1$.

Eg: • $\left(\begin{tikzcd}[row sep=tiny] & 2 \\ 1 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd}\right)$ has a unique weighting: $k^1 = -1$, $k^2 = k^3 = 1$.

• Monoid M has unique weighting: $k = 1/|M|$.

Properties:

- If $\mathbb{A}$ has Möbius inversion then it has a unique weighting, $k^a = \sum_b \mu(a,b)$.
- $\mathbb{A}$ may have 0, 1, or >1 weighting.
- If $\mathbb{A} \simeq \mathbb{B}$ then $\mathbb{A}$ has a weighting $\Leftrightarrow$ $\mathbb{B}$ does.

Componentwise flatness

A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is componentwise flat if $\mathrm{Elts}(X)$ has the following diagram-completion properties:

$$
\begin{tikzcd}
 & \bullet \arrow[dl, dashed] \arrow[dr, dashed] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
 & \bullet &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[d, dashed] \\
\bullet \arrow[d, shift left] \arrow[d, shift right] \\
\bullet
\end{tikzcd}
$$

[Equivalent to:
- X is a sum of flat functors
- $- \otimes X$ preserves pullbacks.]

E.g.:

$\mathbb{A}$	C/w flatness means:
$2 \xleftarrow{u} 1 \xrightarrow{v} 3$	Xu & Xv are injective
monoid M	action is free
$1 \underset{v}{\overset{u}{\rightrightarrows}} 2$	Xu & Xv are injective; $\mathrm{im}(Xu) \cap \mathrm{im}(Xv) = \emptyset$

Result

Thm: Suppose that $\mathbb{A}$ is Cauchy-complete and admits a weighting $k^{\bullet}$. Let $X : \mathbb{A} \to \mathbf{FinSet}$ be componentwise flat. Then

$$|\varinjlim X| = \sum_a k^a |Xa|. \qquad \cdots (*)$$

Proof: By a standard lemma, X is a sum of representables. But the class of functors X satisfying $(*)$

... contains all representables

... is closed under sums. $\square$

E.g.:
- Motivating example: get $|X_2 +_{X_1} X_3| = |X_2| + |X_3| - |X_1|$
- Similarly, general inclusion-exclusion formula
- Free monoid action: $|S/M| = |S| / |M|$.

2. EULER CHARACTERISTIC

Let $\mathbb{A}$ be a finite category.

A coweighting on $\mathbb{A}$ is a weighting on $\mathbb{A}^{op}$.

Lemma: If $k^\bullet$ is a weighting and $k_\bullet$ a coweighting on $\mathbb{A}$, then $\sum_a k^a = \sum_a k_a$. $\square$

E.g.: If $\mathbb{A}$ has Möbius inversion,

$$\begin{array}{c|c} \mu = \begin{pmatrix} \bullet & \cdots & \bullet \\ \vdots & & \vdots \\ \bullet & \cdots & \bullet \end{pmatrix} & \begin{matrix} k^{a_1} \\ \vdots \\ k^{a_n} \end{matrix} \\ \hline \begin{matrix} k_{a_1} & \cdots & k_{a_n} \end{matrix} & \boldsymbol{\chi(\mathbb{A})} \end{array}$$

Defn: $\mathbb{A}$ has Euler characteristic if it admits a weighting and a coweighting. Then

$$\chi(\mathbb{A}) = \sum_a k^a = \sum_a k_a \quad \in \mathbb{Q}$$

for any weighting $k^\bullet$ and coweighting $k_\bullet$.

Examples

- If $\mathbb{A}$ is discrete then $\chi(\mathbb{A}) = |\mathrm{ob}\,\mathbb{A}|$.

- If $\mathbb{A}$ contains no non-trivial endos then

$$\chi(B\mathbb{A}) = \chi(\mathbb{A})$$

where $B\mathbb{A} = |N\mathbb{A}|$ is classifying space of $\mathbb{A}$.

- If G is a circuit-free directed graph then

$$\chi(FG) = |G_0| - |G_1|$$

where FG is free cat on G.

- (Rota) If $\mathbb{A}$ is a poset then

$$\chi(\mathbb{A}) = \sum_n (-1)^n \, |\{\text{chains } a_0 < \cdots < a_n\}|.$$

Similar formulas cover other classes of cat.

- If M is a monoid then $\chi(M) = 1/|M|$.

- For groupoids, agrees with Baez-Dolan.

Properties

- If $\mathbb{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbb{B}$ then $\chi(\mathbb{A}) = \chi(\mathbb{B})$
- If $\mathbb{A} \simeq \mathbb{B}$ then $\chi(\mathbb{A}) = \chi(\mathbb{B})$
- If $\mathbb{A}$ has a 0 or a 1 then $\chi(\mathbb{A}) = 1$
- $\chi(\mathbb{A}^{op}) = \chi(\mathbb{A})$
- $\chi(\sum \mathbb{A}_i) = \sum \chi(\mathbb{A}_i)$, $\quad \chi(\prod \mathbb{A}_i) = \prod \chi(\mathbb{A}_i)$
- Fibration formula:

  if $X : \mathbb{A} \longrightarrow \mathbf{Cat}$ and $k^\bullet$ is a weighting on $\mathbb{A}$ then

$$\chi(\mathrm{Elts}(X)) = \sum_a k^a \chi(Xa).$$

Further examples

- Let $X : M \longrightarrow \mathbf{Set}$ be an action of a monoid M on a set S. Write $\mathrm{Elts}(X) = S/\!/M$. Then
$$\chi(S/\!/M) = |S|/|M|.$$

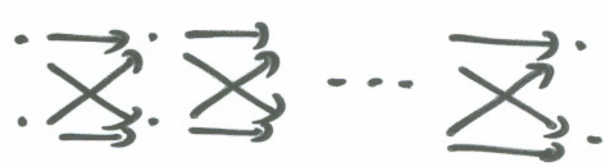
- Rota's theory: have comm square

$$\begin{tikzcd}
\{\text{triangulated manifolds}\} \arrow[r] \arrow[d, "\text{forget}"'] & \{\text{posets}\} \arrow[d, "\chi"] \\
\{\text{manifolds}\} \arrow[r, "\chi"'] & \mathbb{Z}
\end{tikzcd}$$

This theory: have comm square

$$\begin{tikzcd}
\{\text{triangulated orbifolds}\} \arrow[r, "\mathrm{MP}"] \arrow[d, "\text{forget}"'] & \{\text{categories}\} \arrow[d, "\chi"] \\
\{\text{orbifolds}\} \arrow[r, "\chi"'] & \mathbb{Q}.
\end{tikzcd}$$

- Let $\mathbb{S}^{-1} = \emptyset$, $\quad \mathbb{S}^n = \mathrm{Elts}\left(\begin{tikzcd}[row sep=small] & \mathbf{1} \\ \mathbb{S}^{n-1} \arrow[ur] \arrow[dr] & \\ & \mathbf{1} \end{tikzcd} \right)$

$=$ 

$$\chi(\mathbb{S}^n) = \chi(\mathbf{1}) + \chi(\mathbf{1}) - \chi(\mathbb{S}^{n-1})$$
$$= 2 - \chi(\mathbb{S}^{n-1})$$

So $\chi(\mathbb{S}^n) = 1 + (-1)^n$.

Conclusion

This is the right Euler characteristic for categories (at least, when defined).

Desire

Find a universal property of the Euler characteristic of categories, à la Schanuel.