On the reflection and the coreflection of categories over a base in discrete fibrations

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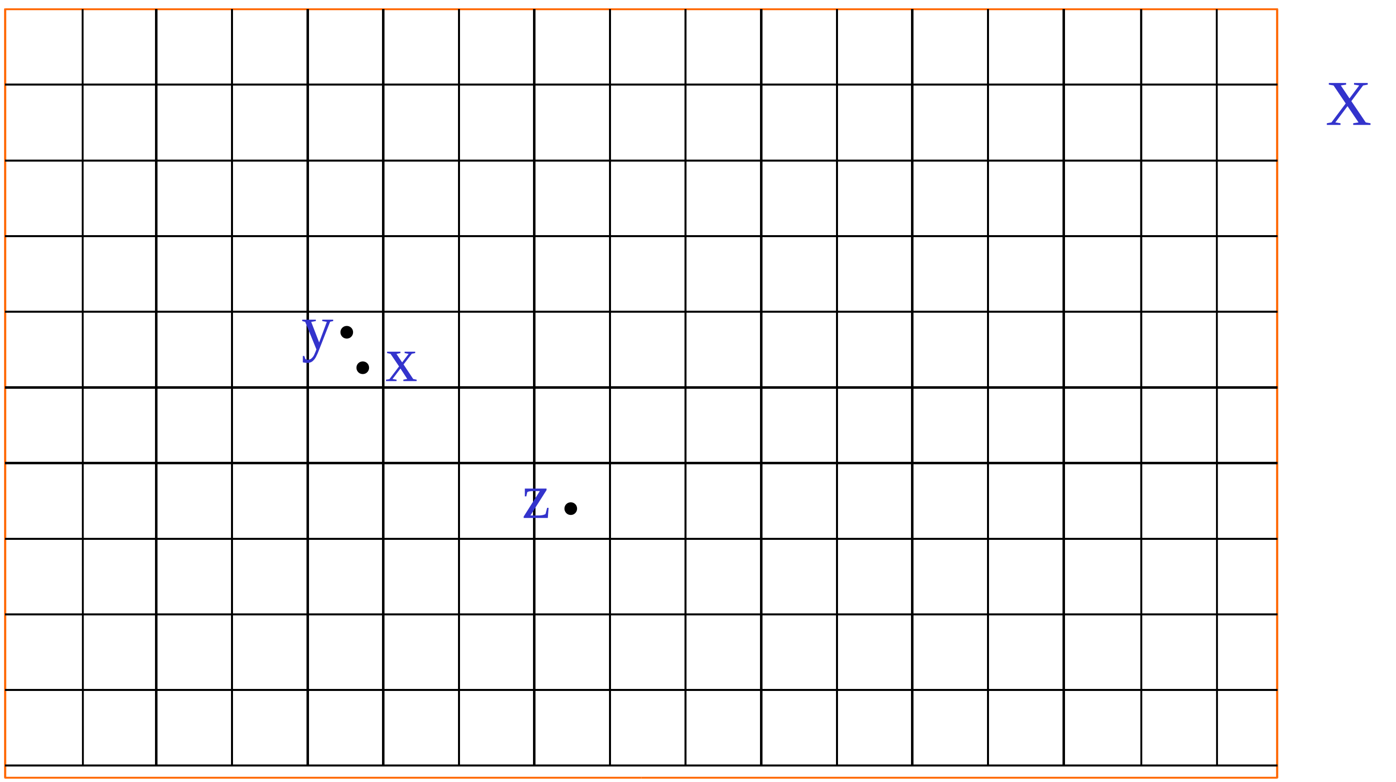
Various aspects

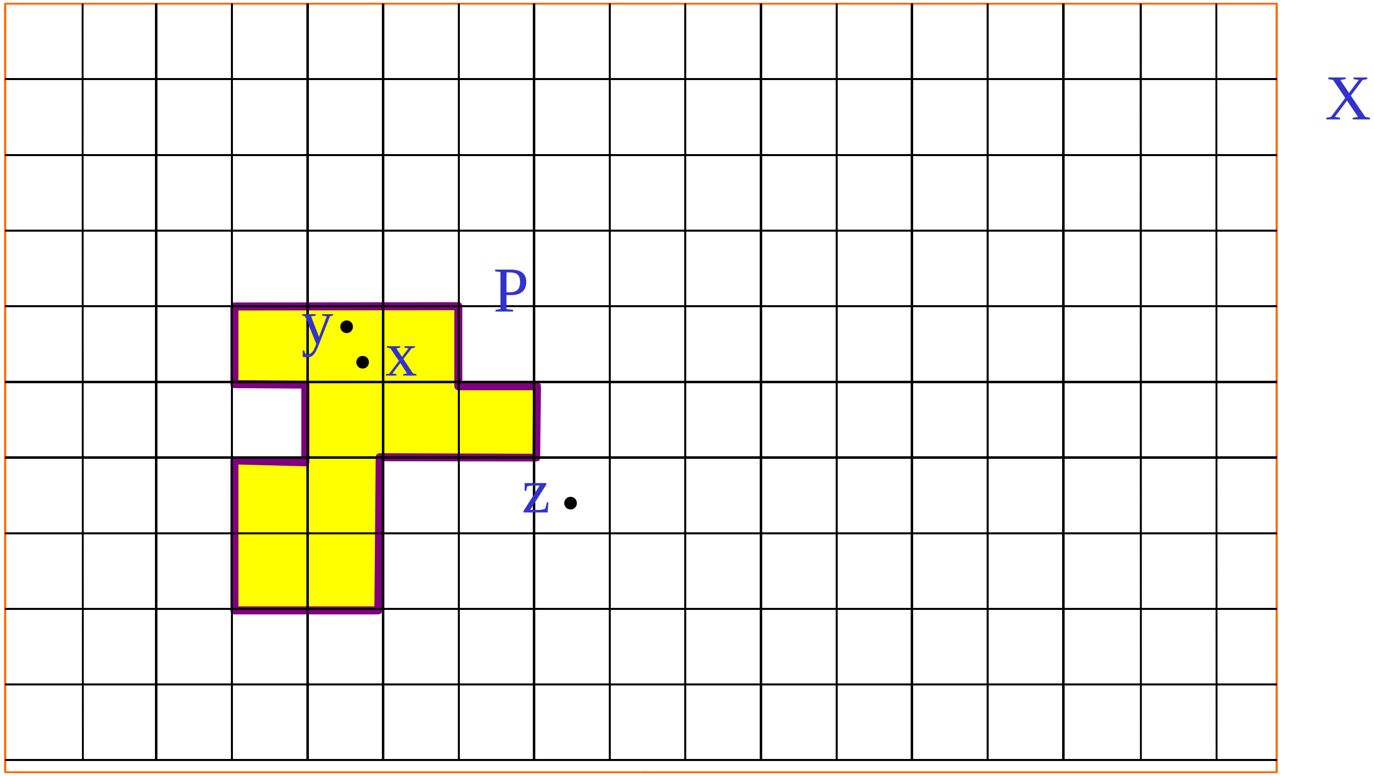
of two "dual" formulas

We begin by illustrating the formulas in the two-valued context.



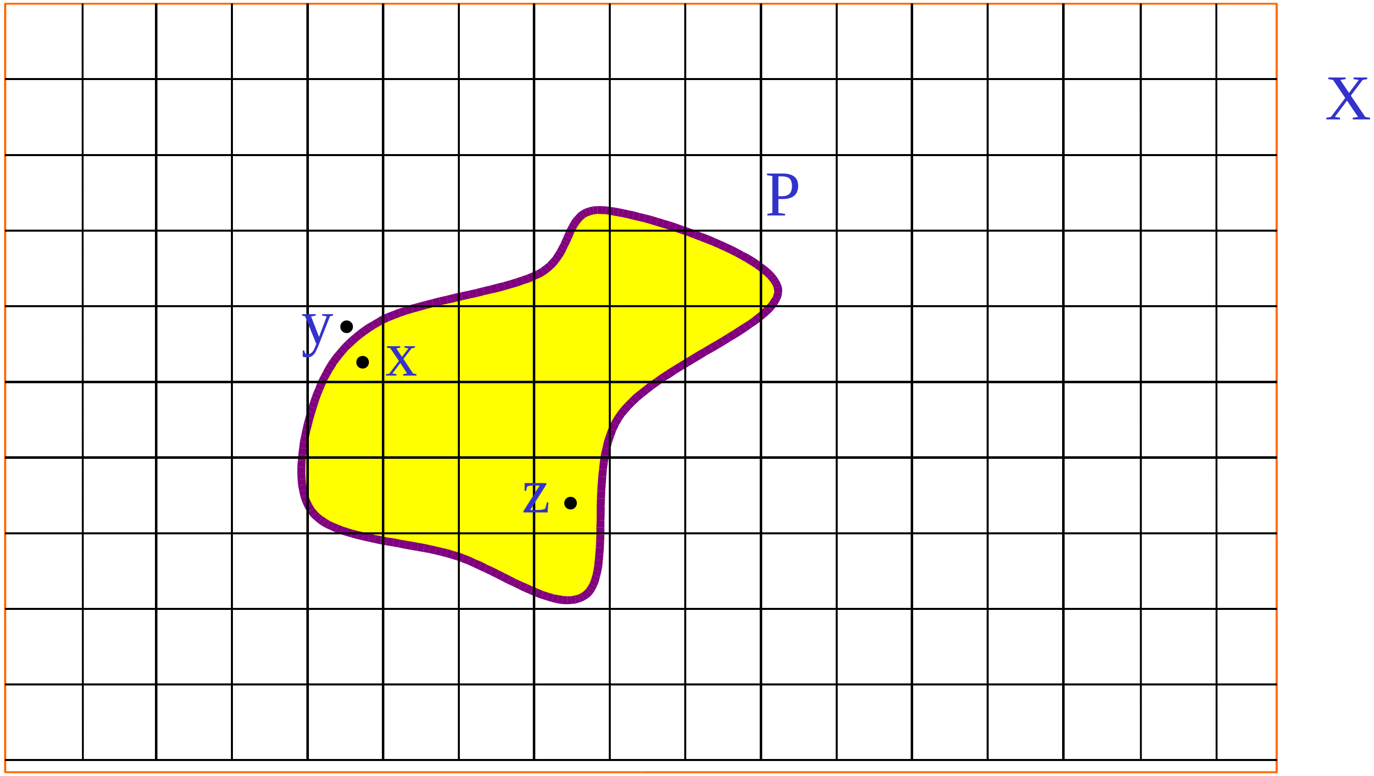
Consider an equivalence relation on a set X

x ~ y   iff   they are in the same square



Some subsets P of X are closed with respect to ~

$$\boxed{x \in P \quad \text{and} \quad x \sim y \qquad \Longrightarrow \qquad y \in P}$$



Some other subsets P of X are not closed

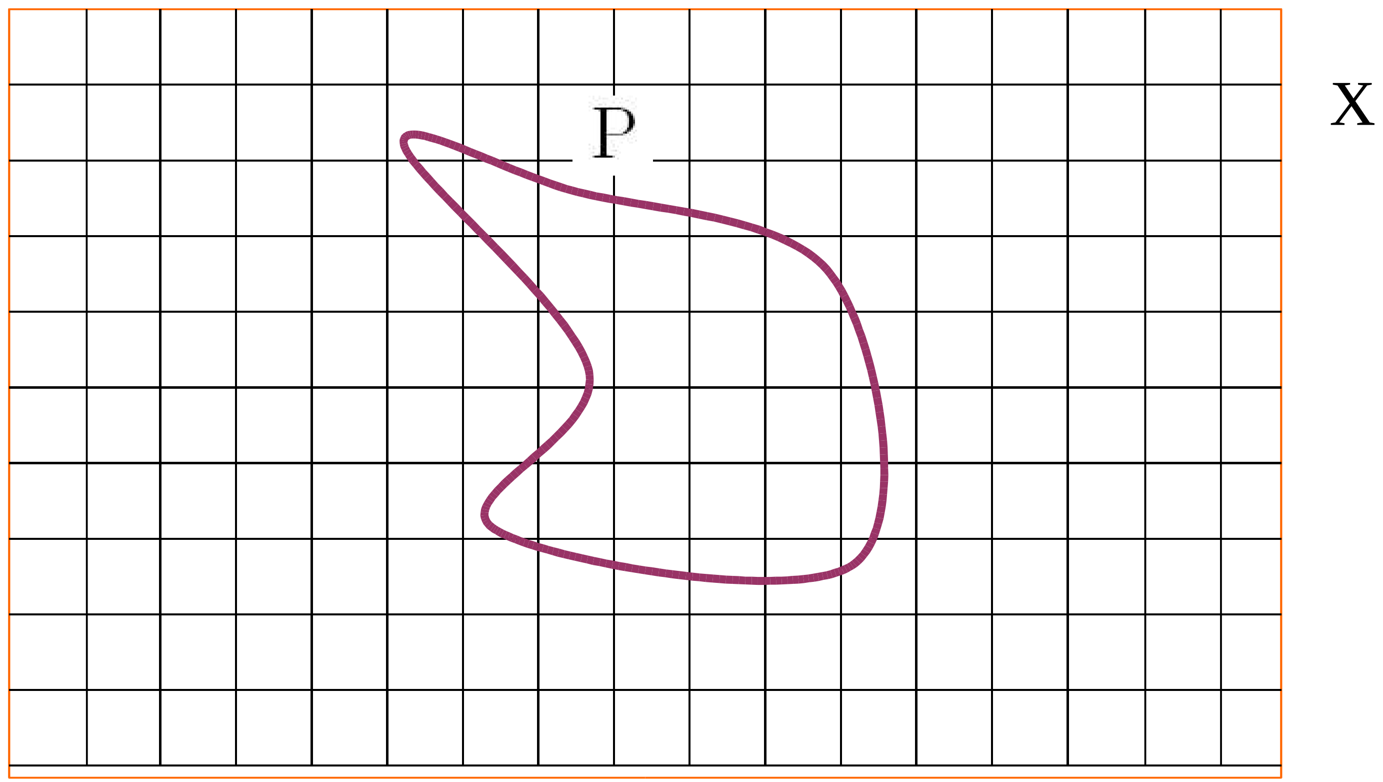
So we have the inclusion

$$\boxed{i : \overline{\mathcal{P}}\mathrm{X} \hookrightarrow \mathcal{P}\mathrm{X}}$$

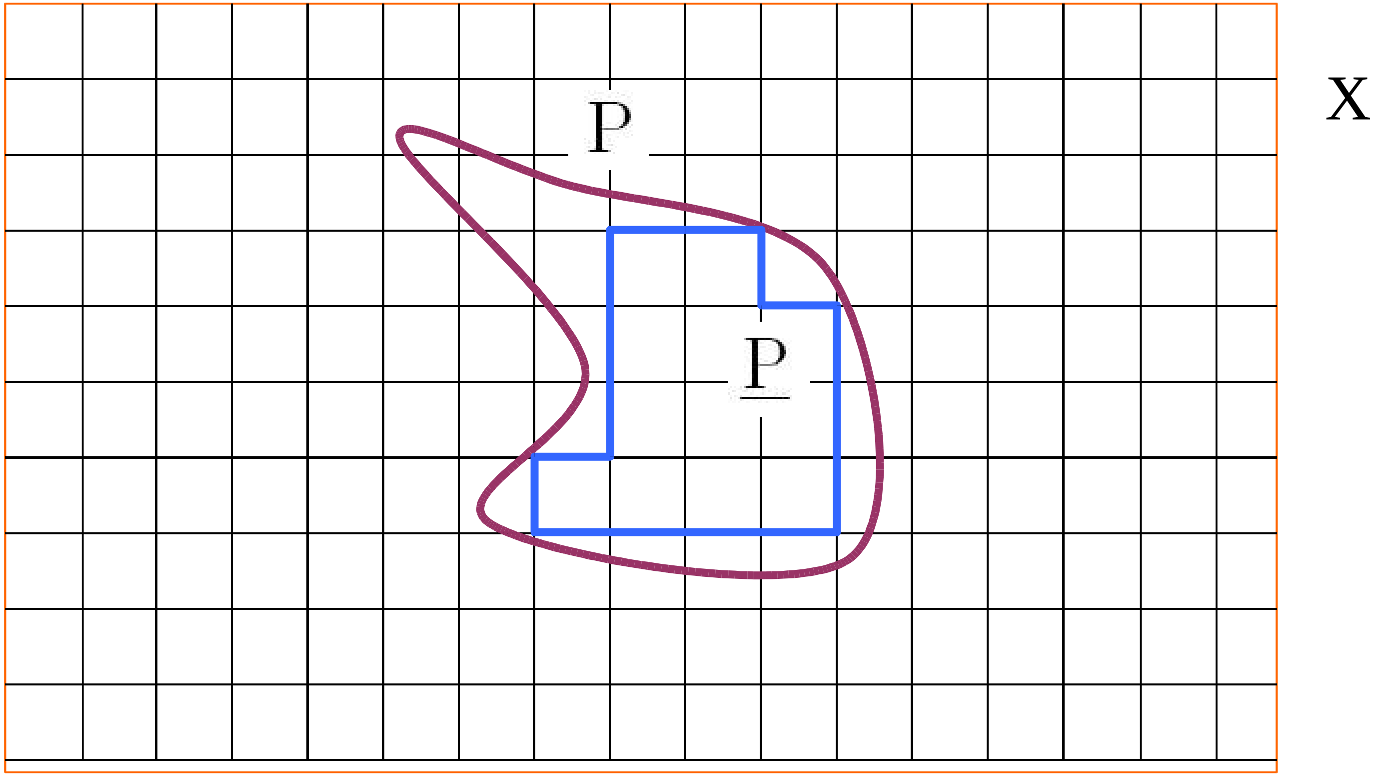
of closed parts in all the parts of X.

Such an inclusion has both left and right adjoints

$$\boxed{\overline{(-)} \dashv i \dashv \underline{(-)}}$$



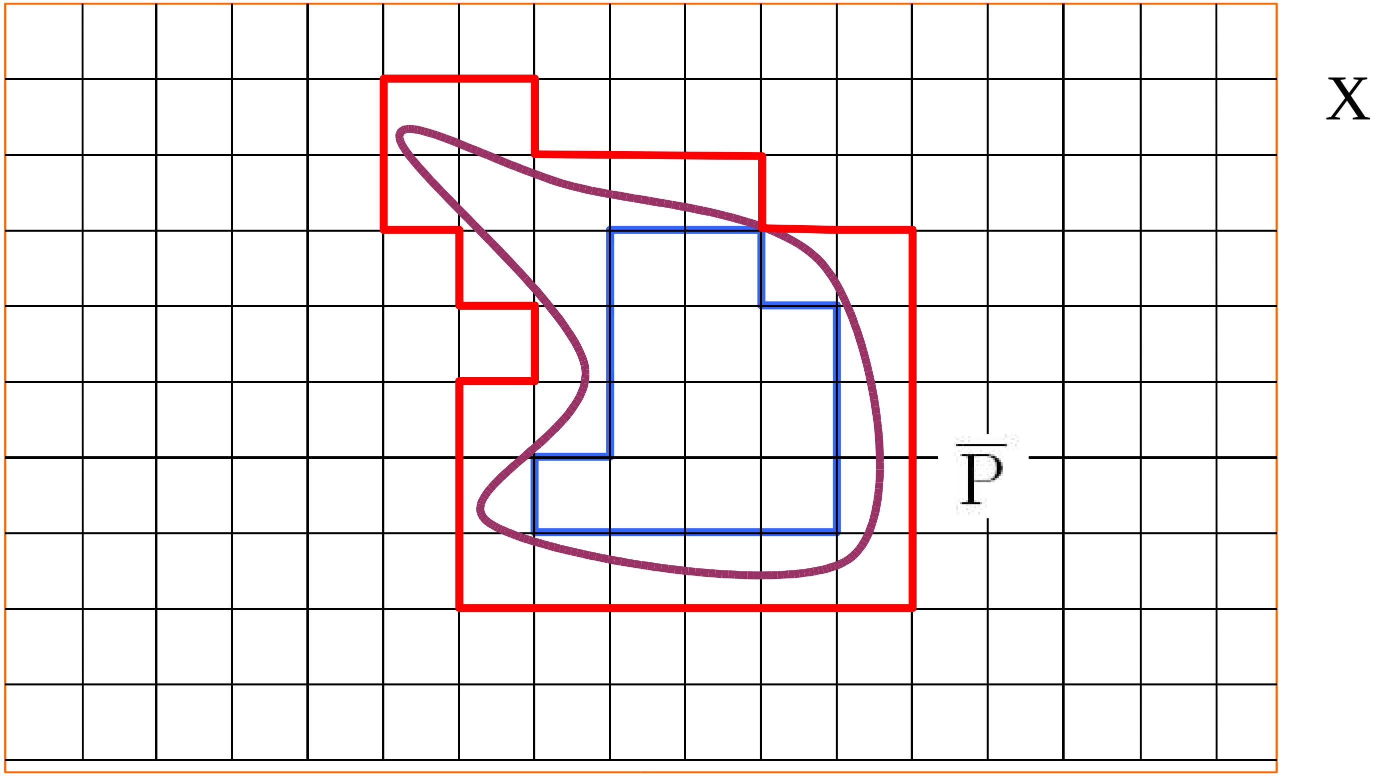
Any subset P of X (only outline drawn)



Any subset P of X

has a "best" inner approximation $\boxed{\underline{P}}$

(its coreflection)

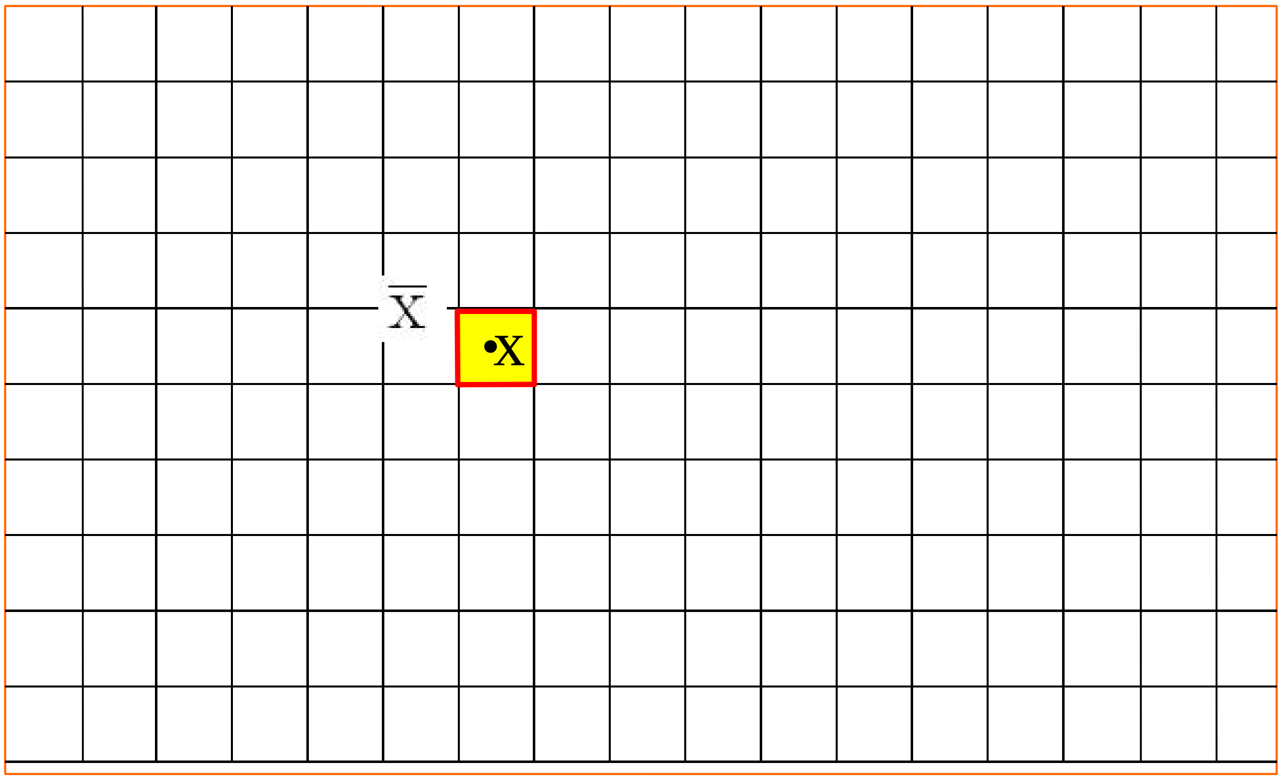


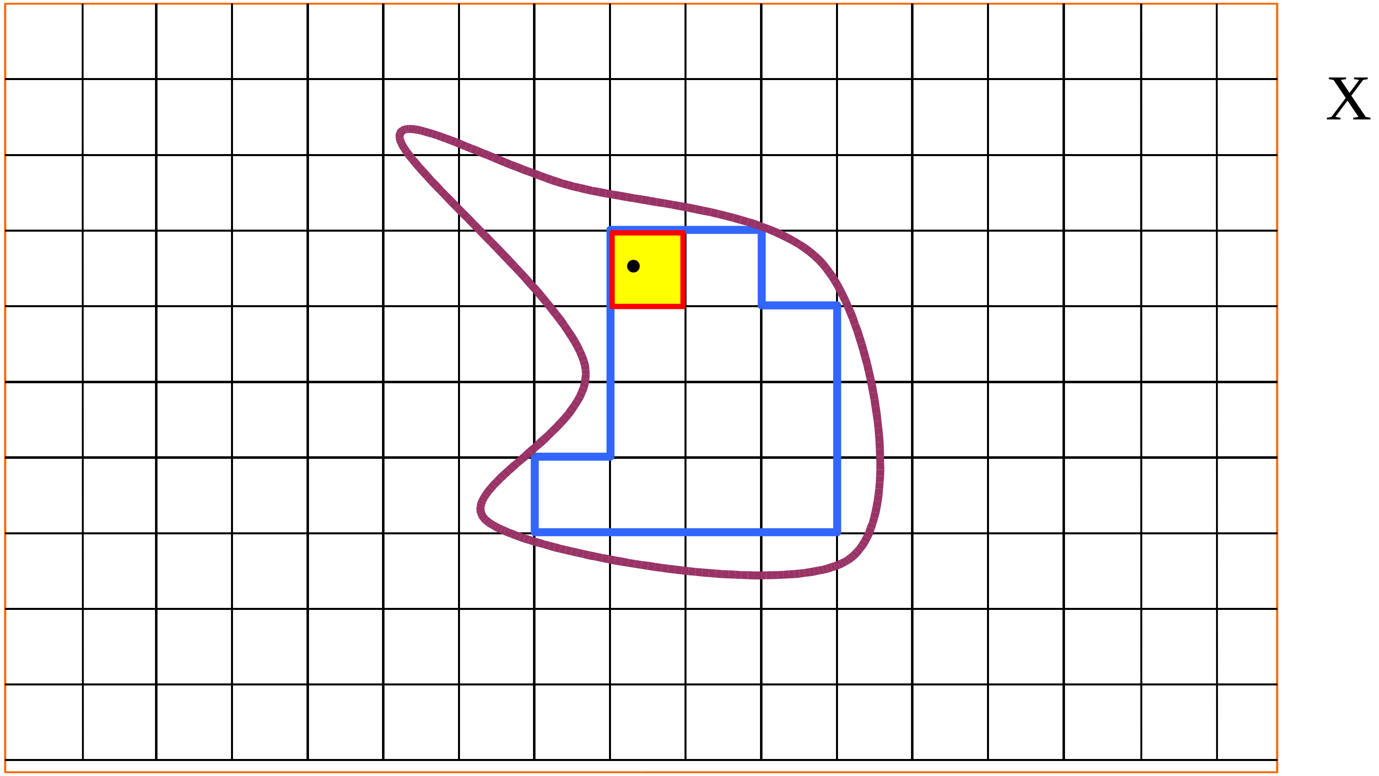
Any subset P of X

has a "best" inner approximation $\underline{P}$

and a "best" outer one $\boxed{\overline{P}}$ (its reflection)

Note in particular that the reflection of a point x is the square of x:

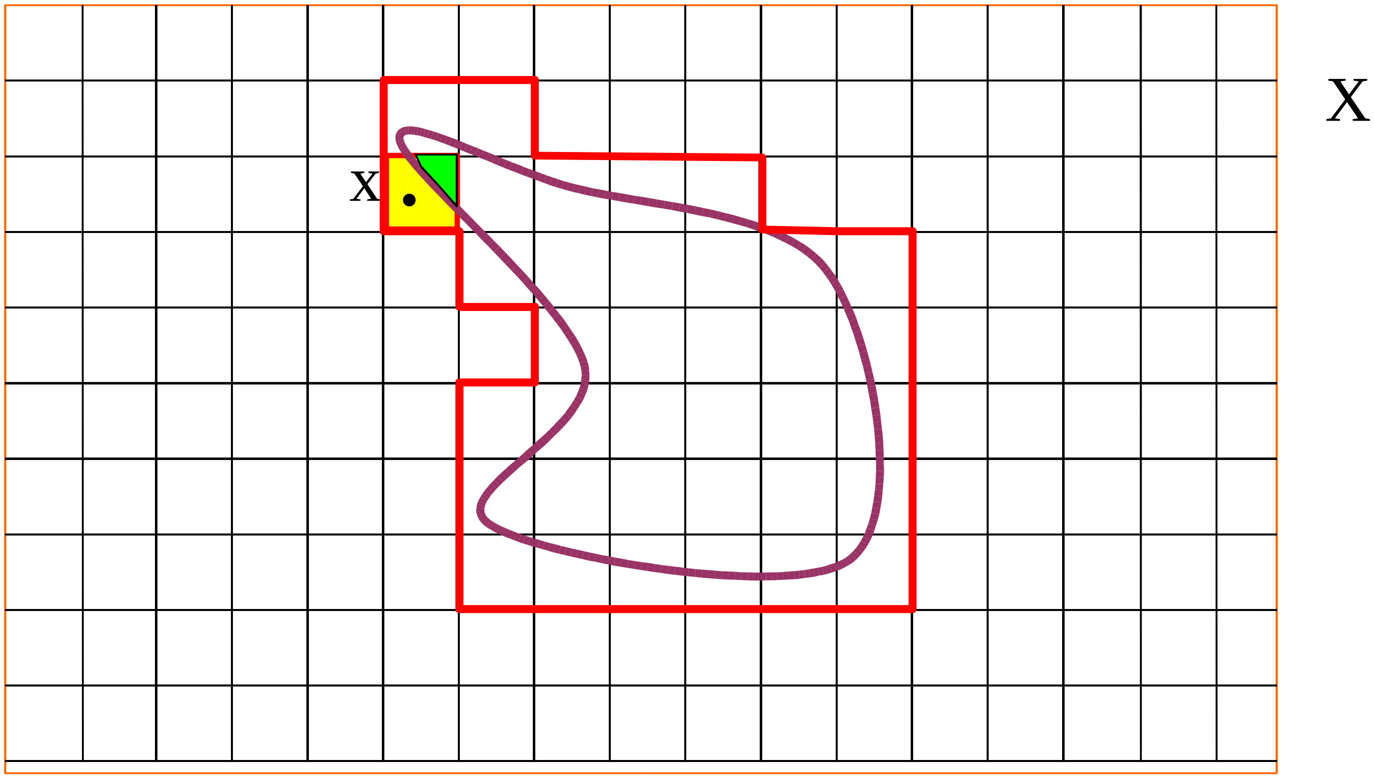




For the coreflection we have the formula:

$$\boxed{x \in \underline{P} \quad \Leftrightarrow \quad \overline{x} \subseteq P}$$

that is, a point is in the coreflection of P iff its square is included in P.



“Dually”, for the reflection we have the formula:

$$\boxed{x \in \overline{P} \quad \Leftrightarrow \quad \overline{x} \pitchfork P}$$

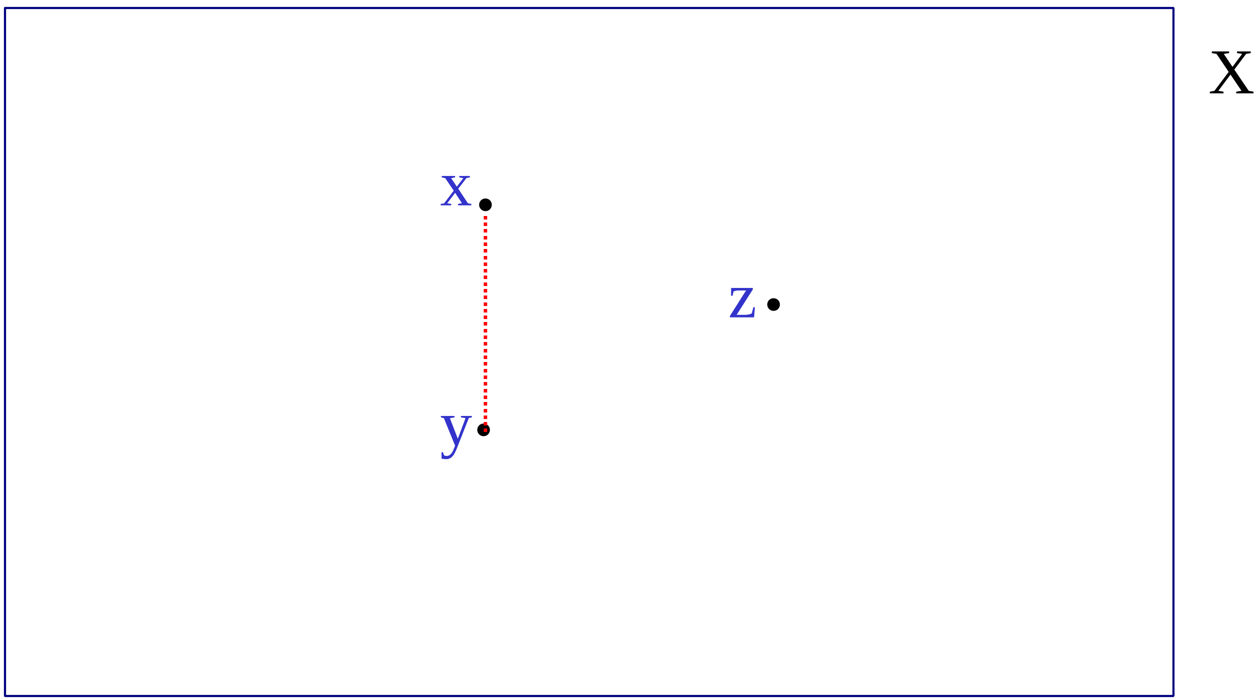
that is, a point is in the reflection of P iff

its square “meets” P (they have non void intersection)

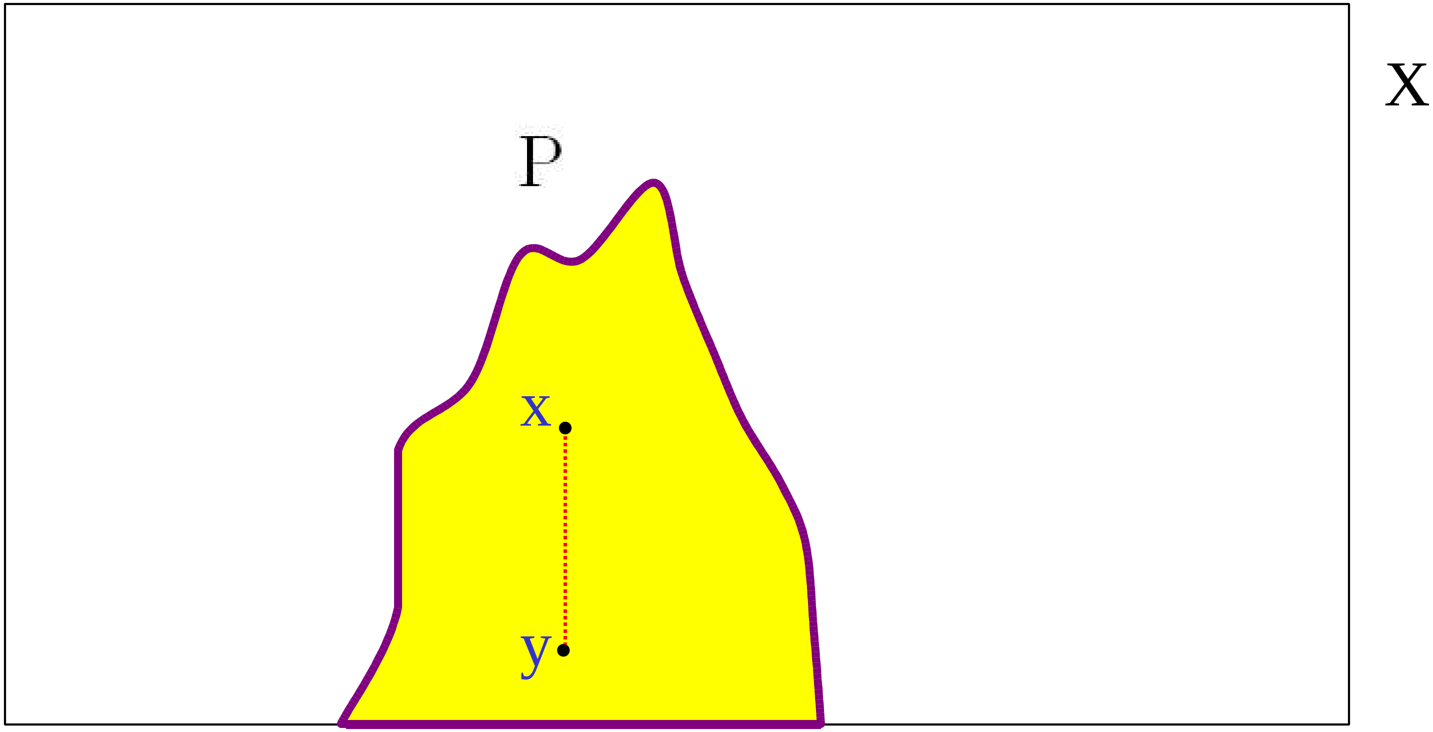
Now let’s drop the symmetry condition:

instead of an equivalence relation, consider an arbitrary poset.

For example, on a part X of the plane consider the following order:

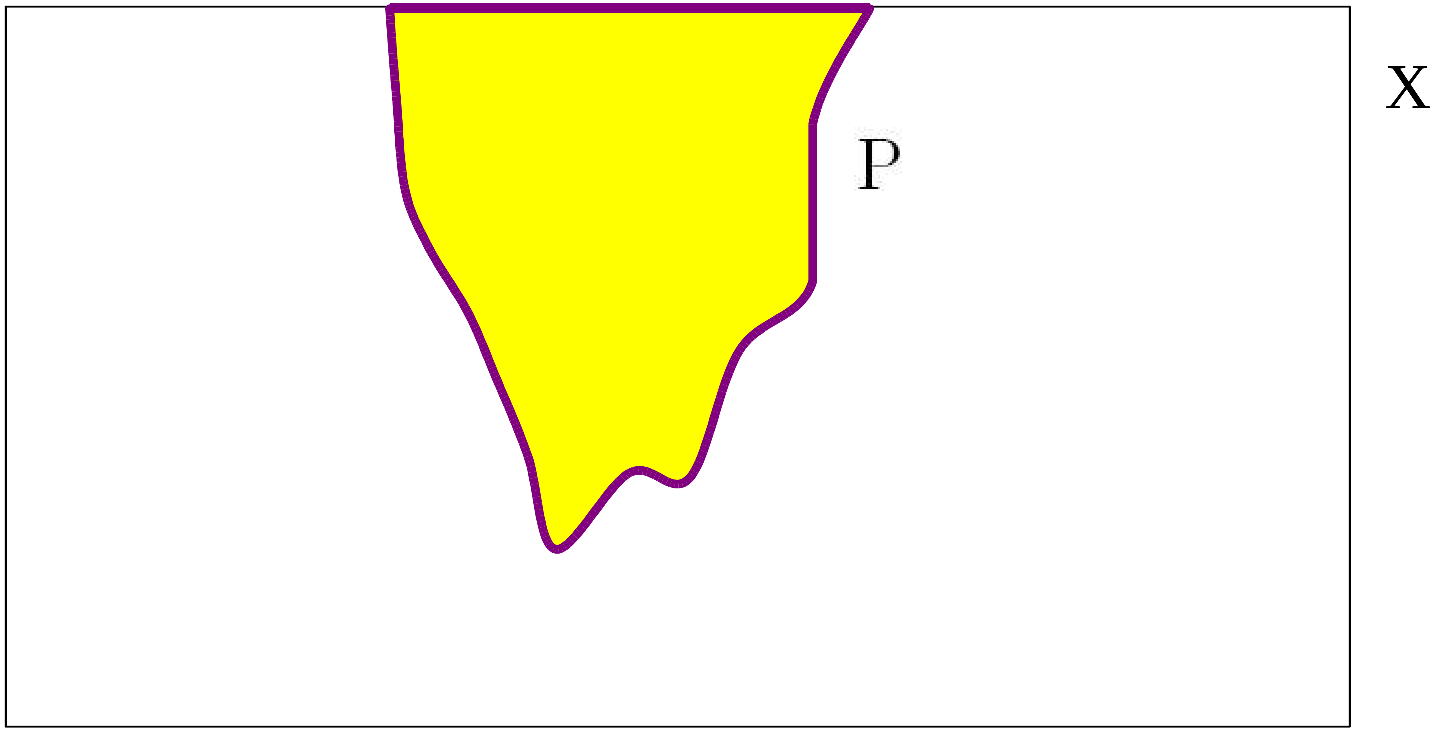


$y \leq x$ iff y is exactly below x



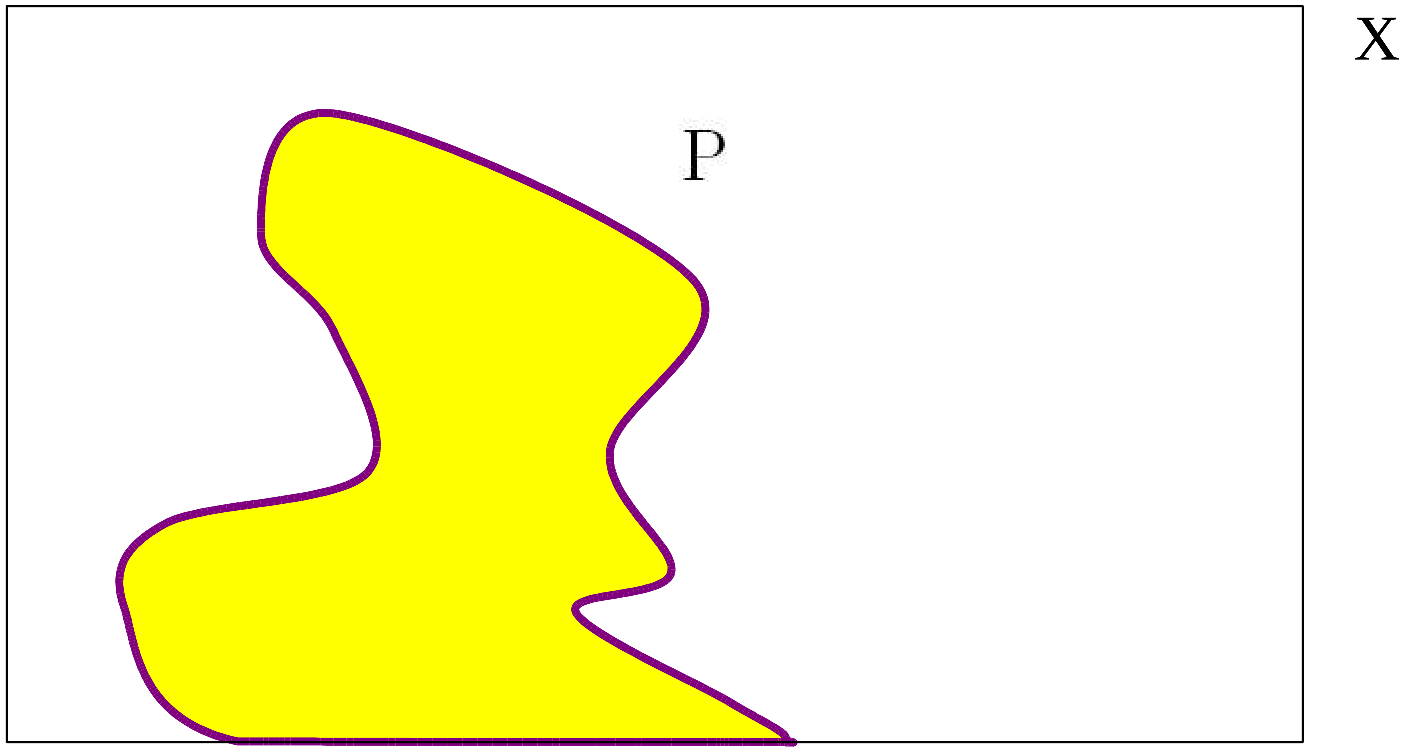
Some subsets P of X are lower sets, that is donward closed:

$$\boxed{x \in P \quad \text{and} \quad y \leq x \qquad \Longrightarrow \qquad y \in P}$$



… some subsets P of X are upper sets, that is upward closed:

$$\boxed{x \in P \quad \text{and} \quad x \leq y \qquad \Longrightarrow \qquad y \in P}$$



Some other subsets P of X are not upper nor lower sets.

So we have the inclusions

$$\boxed{i : \overleftarrow{\mathcal{P}} X \hookrightarrow \mathcal{P} X} \qquad \boxed{j : \overrightarrow{\mathcal{P}} X \hookrightarrow \mathcal{P} X}$$

of lower and upper parts in all the parts of X.

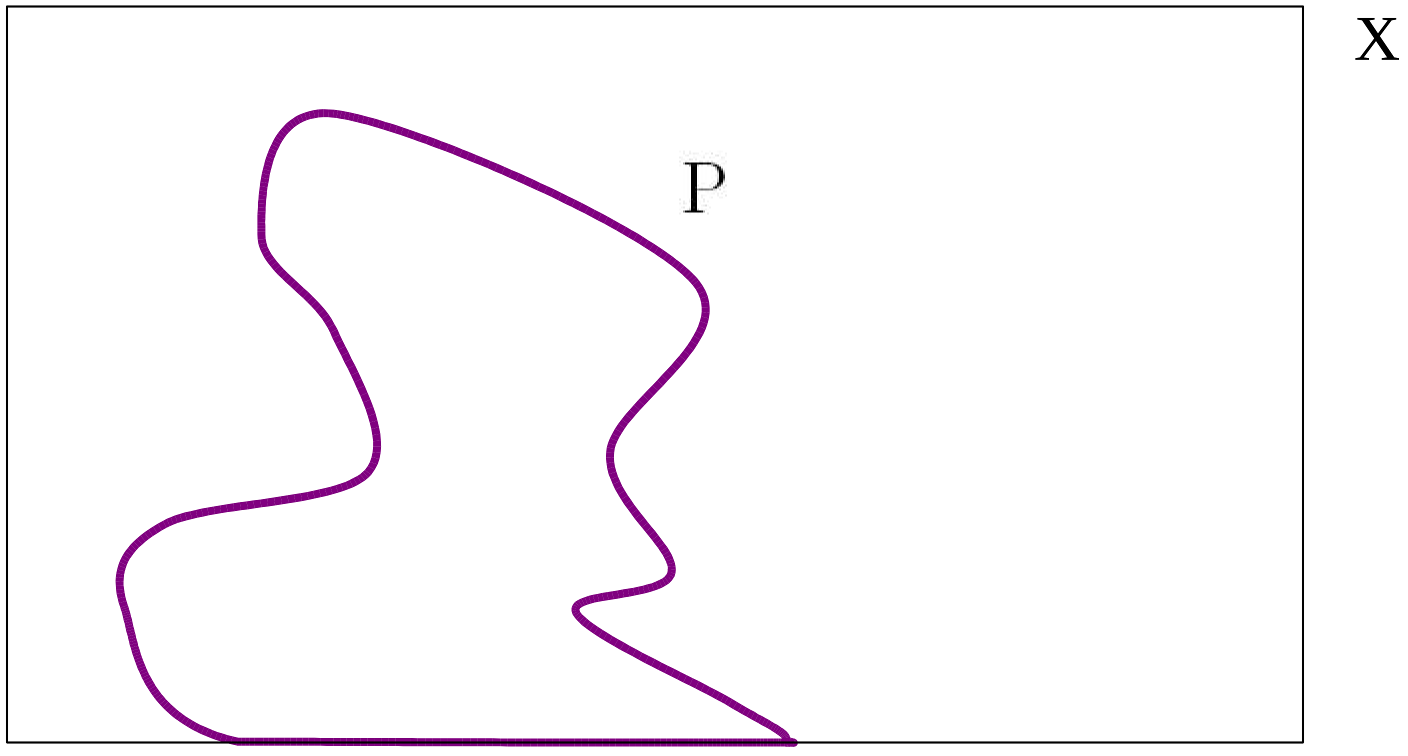
Such inclusions have both left and right adjoints

$$\boxed{\uparrow(-) \dashv i \dashv (-)\uparrow} \qquad \boxed{\downarrow(-) \dashv j \dashv (-)\downarrow}$$

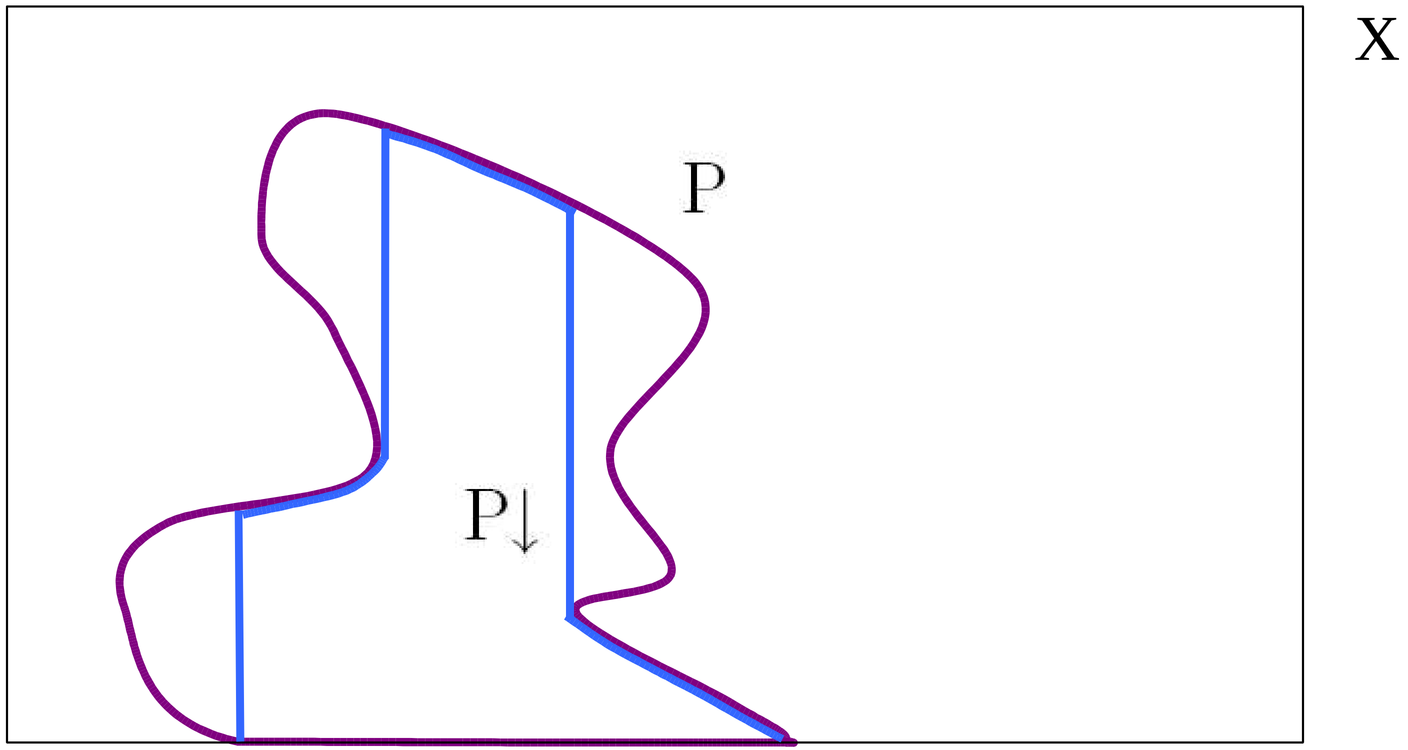
Let’s consider the reflection and coreflection

in lower sets

(the case of upper sets is of course specular)

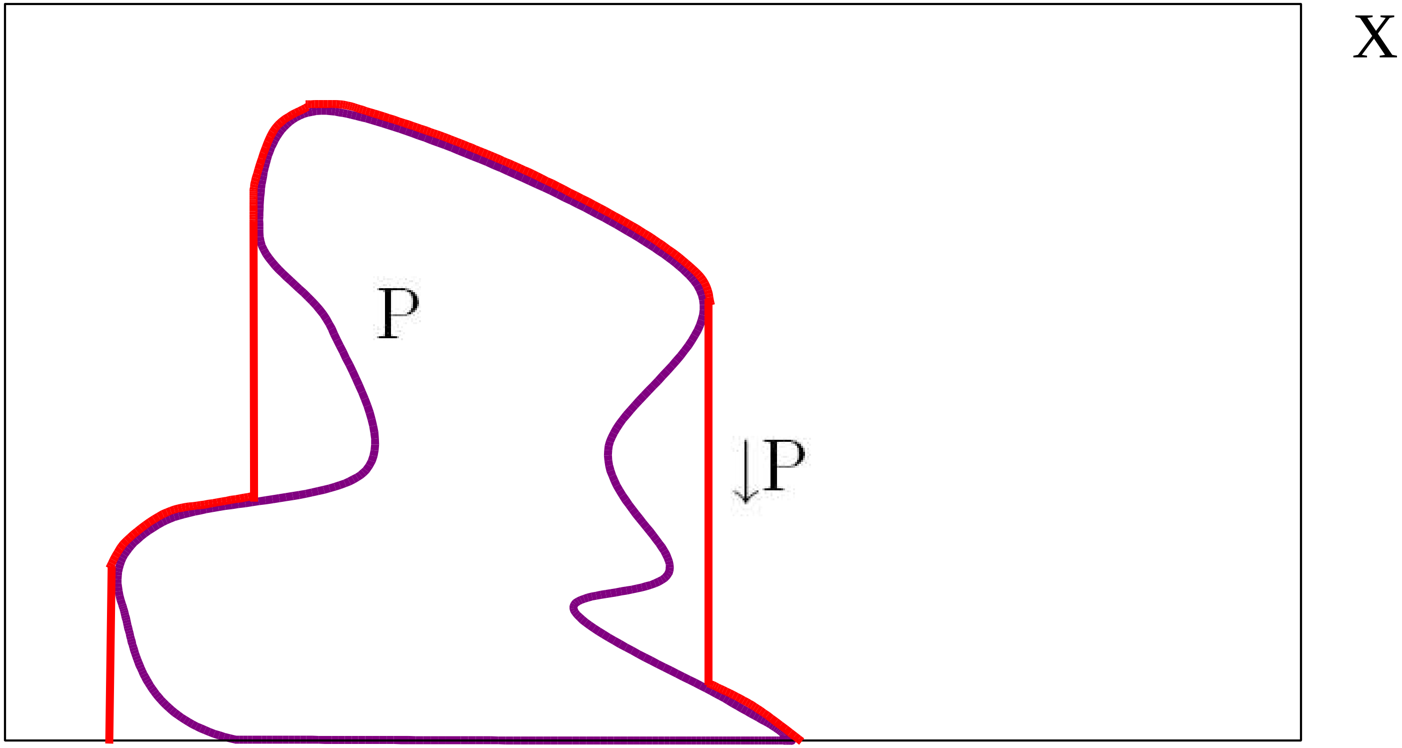


Any subset P of X (only outline drawn)



Any subset P of X (only outline drawn)

has a “best” inner approximation $\boxed{P\downarrow}$

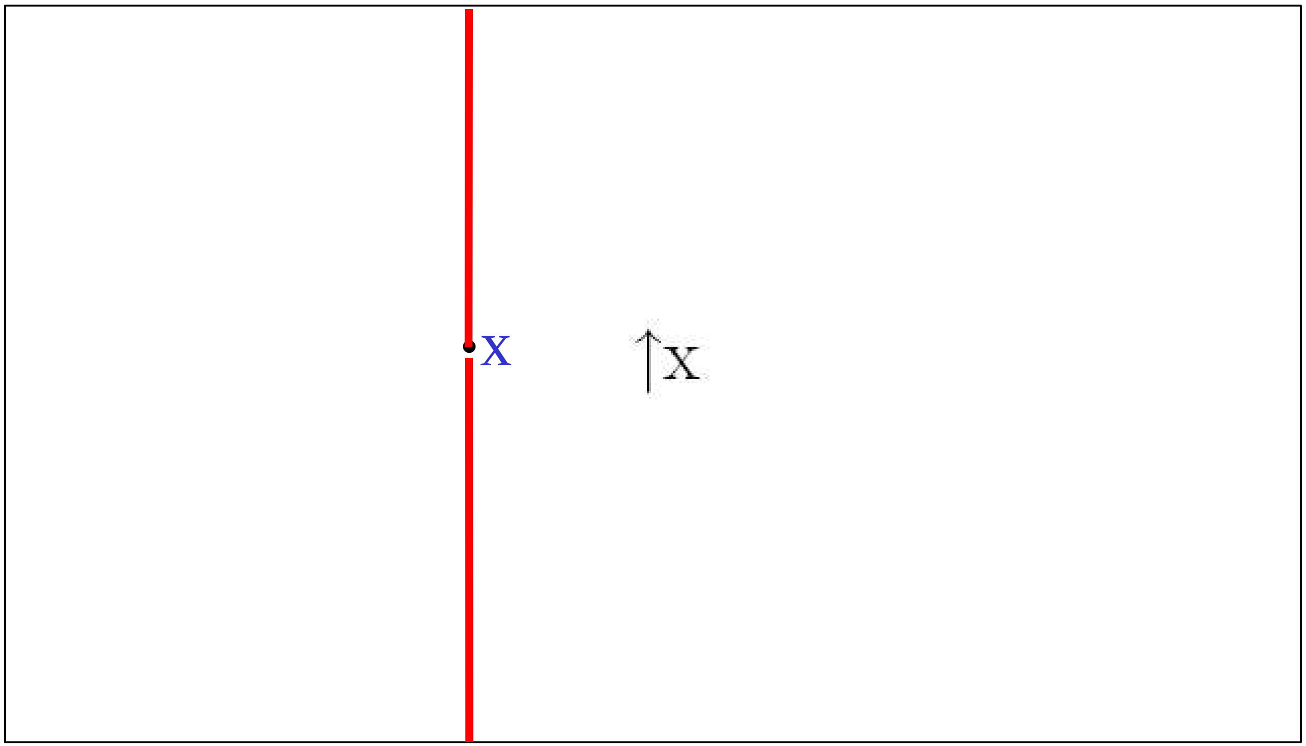


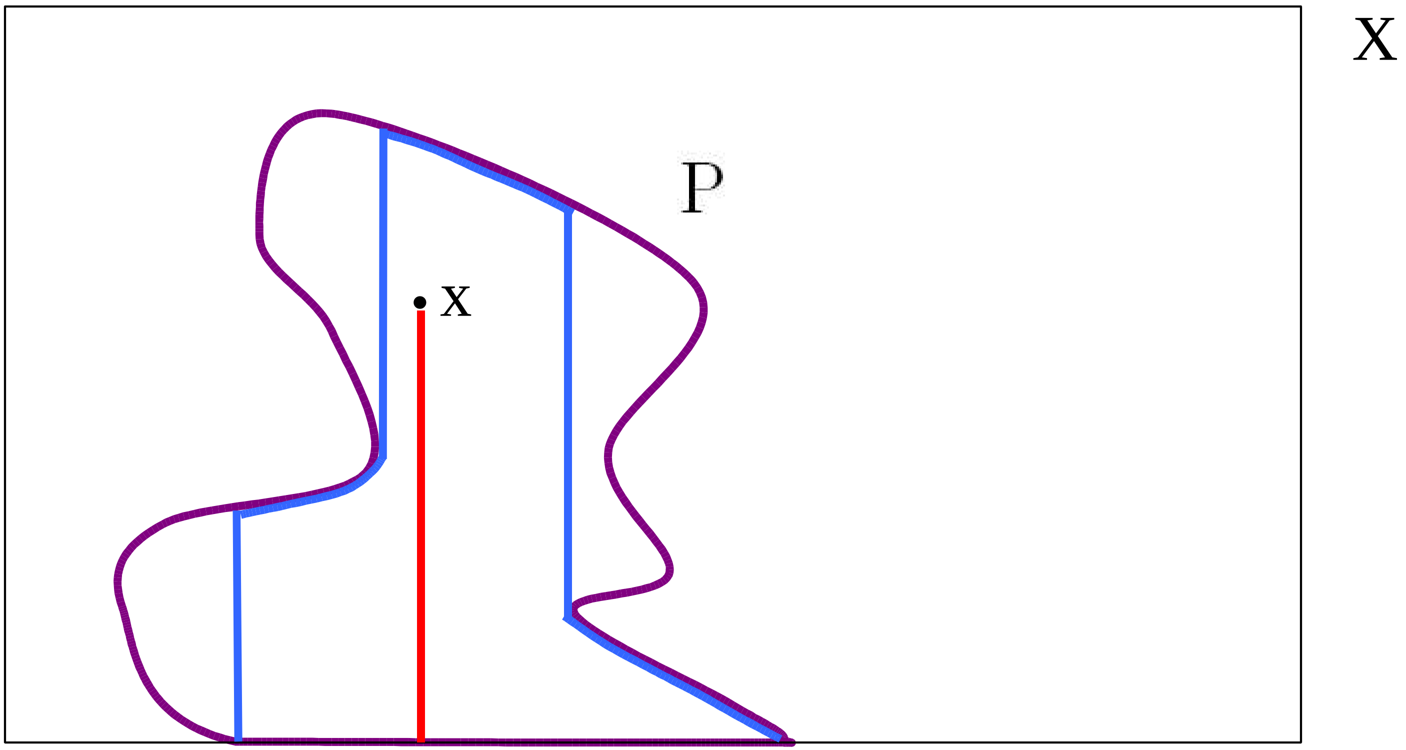
Any subset P of X (only outline drawn)

has a "best" inner approximation $P\downarrow$

and a "best" outer one $\boxed{\downarrow P}$

Note in particular that the lower and upper reflections of a point x are:

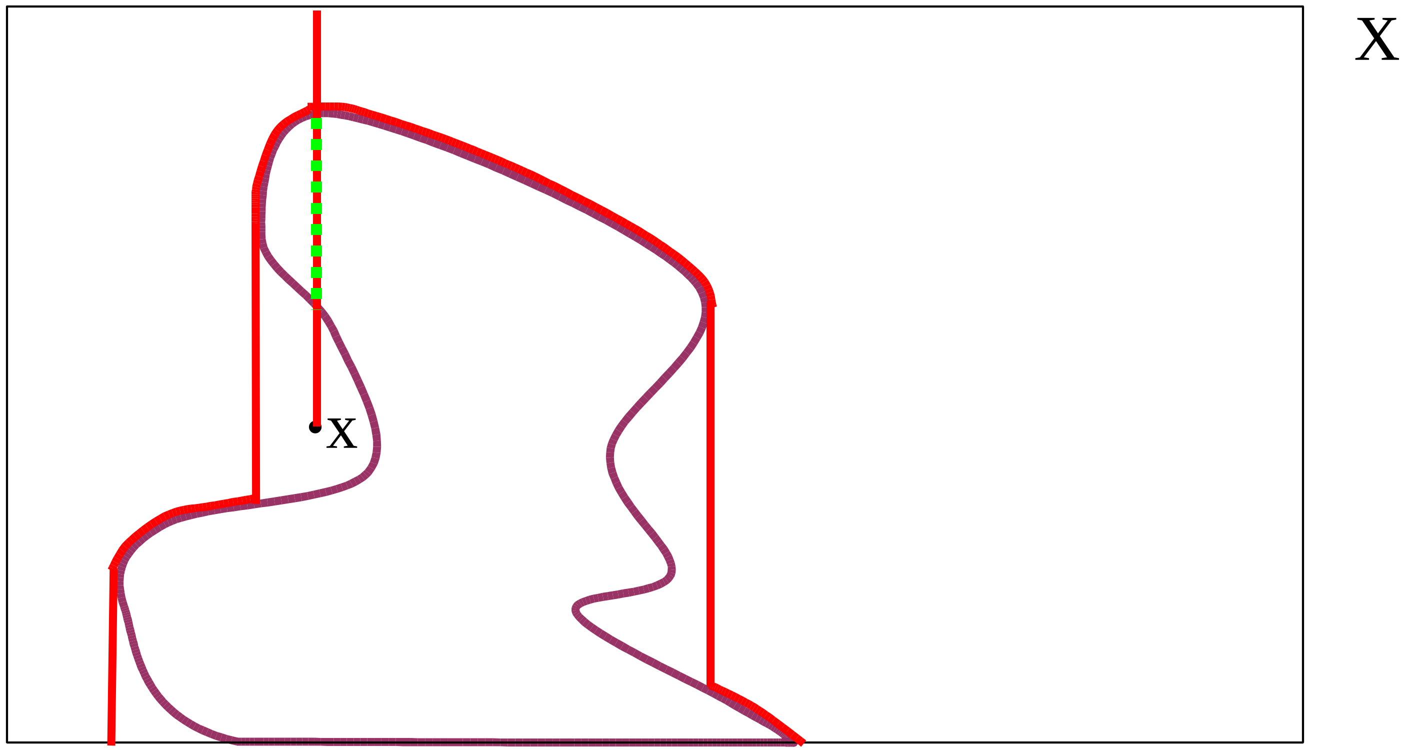




For the coreflection, we have the formula:

$$\boxed{x \in P{\downarrow} \quad \Leftrightarrow \quad {\downarrow}x \subseteq P}$$

that is, a point is in the lower coreflection of P iff its lower reflection is included in P.



“Dually”, for the reflection we have the formula:

$$\boxed{x \in \downarrow P \quad \Leftrightarrow \quad \uparrow x \pitchfork P}$$

that is, a point is in the lower reflection of P iff its upper reflection meets P.

In order to pass to the set-valued context, we need to look more closely to the "meets" operator $\pitchfork$:

let X be a bounded poset (that is with top $\top$ and bottom $\perp$)

and let $\mathbf{2} = \{\mathrm{true}, \mathrm{false}\}$ be the truth values poset.

We have the following chain of adjoint functors (poset morphisms)

$$\boxed{\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : X \to \mathbf{2}}$$

where $\Gamma^*(\mathrm{true}) = \top$ nd $\Gamma^*(\mathrm{false}) = \perp$

$\Gamma_!$ and Γ_* are the "non void" and "full" predicates:

$$\boxed{\Gamma_! x \quad \text{is false} \quad \text{iff} \quad x \leq \bot}$$

$$\boxed{\Gamma_* x \quad \text{is true} \quad \text{iff} \quad \top \leq x}$$

So, if X is also a meet semilattice, we have the following obvious definition of the "meets" predicate:

$$\boxed{x \pitchfork y \quad \Leftrightarrow \quad \Gamma_!(x \wedge y)}$$

Now, let’s jump from the two-valued into the set-valued context, replacing posets with categories:

$\boxed{\text{Which is the correspective of the meets operator?}}$

For many categories X, there are analogous adjoint functors

$$\boxed{\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : X \to \mathbf{Set}}$$

the “components”, the “discrete” and the “points” functors.

Note that they are uniquely determined, since Γ_* is forced

to be the functor represented by the terminal object of X:

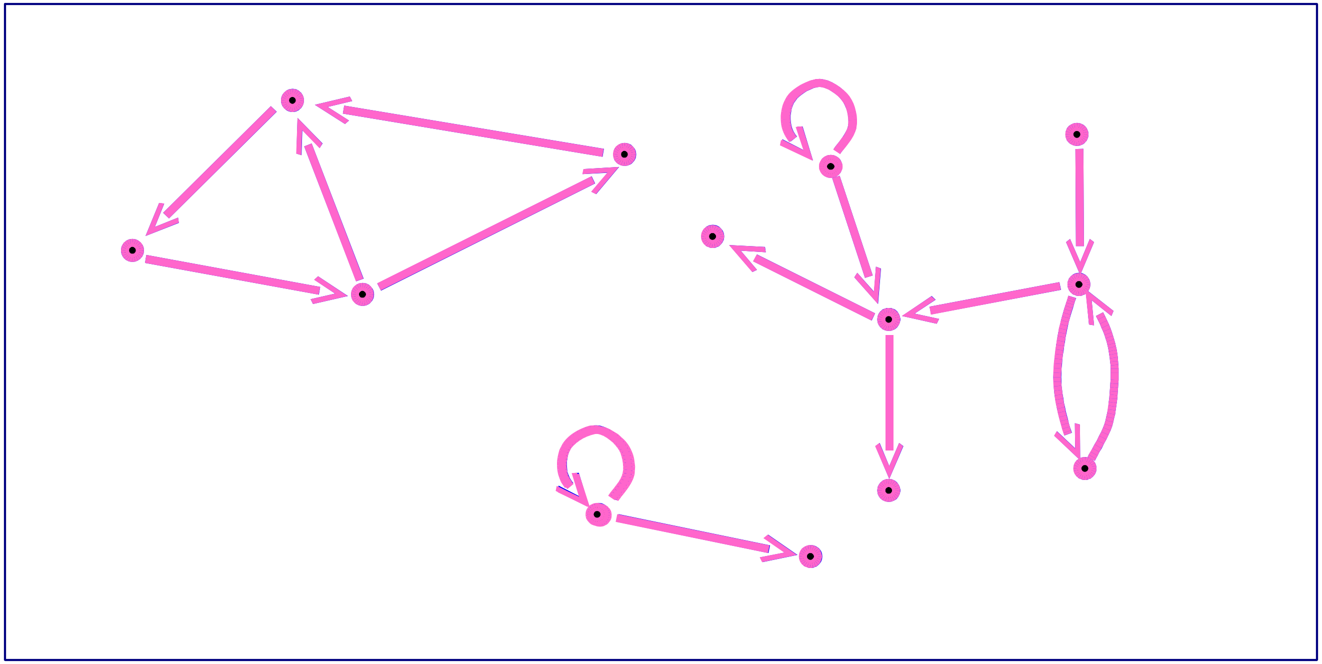
$$\Gamma_* = X(1, -)$$

Typically, if $\boxed{\mathcal{G}}$ is the category of (directed irreflexive) graphs, we have:

$$\boxed{\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : \mathcal{G} \longrightarrow \mathbf{Set}}$$

The components and the points of a graph are given by the coequalizer and the equalizer of the domain and codomain maps, respectively.

In general, for any presheaf category, Γ_* and $\Gamma_!$ give the limit and the colimit respectively.



This graph has three components and two points.

If X has products, we can generalize naturally the “meets” operator to the set-valued setting, obtaining a “ten” functor:

$$\boxed{x \pitchfork y \quad \Leftrightarrow \quad \Gamma_!(x \wedge y)}$$

two-valued

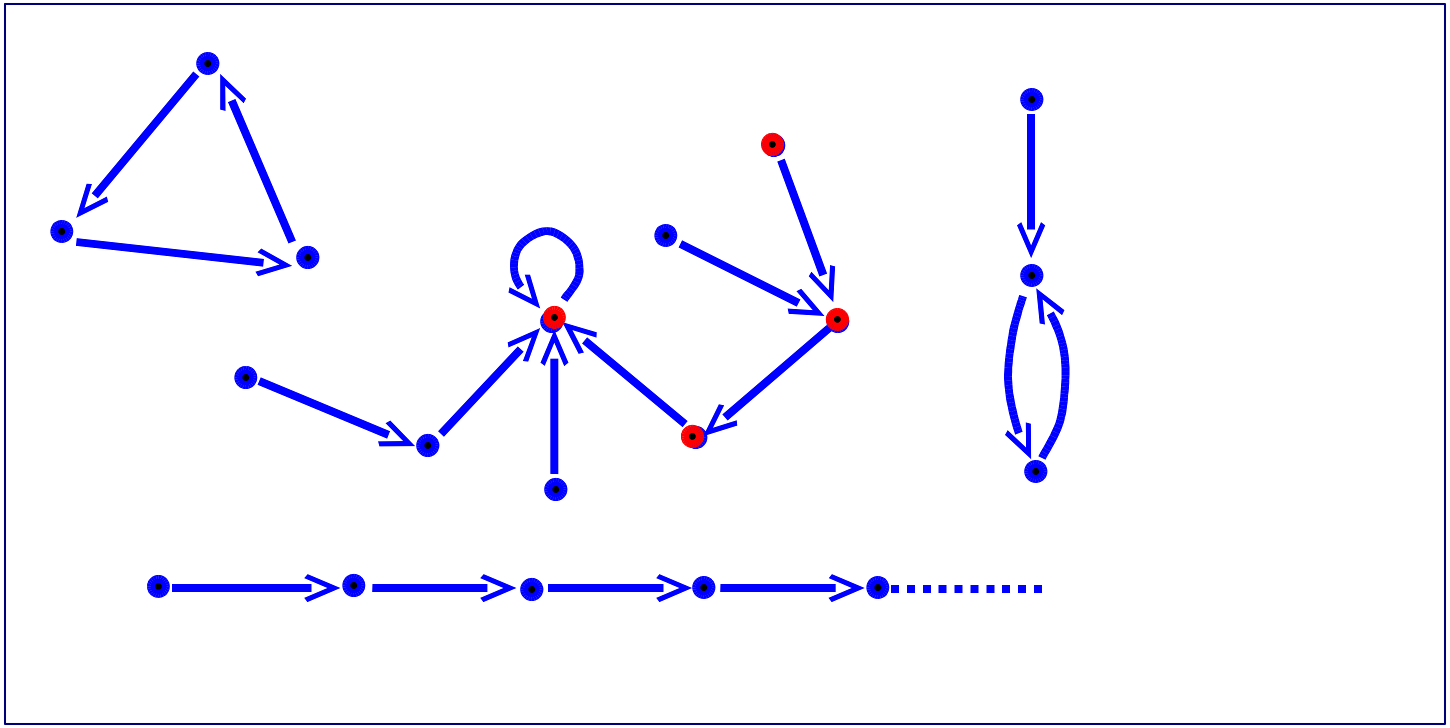
$$\boxed{\mathrm{ten}(x, y) = \Gamma_!(x \times y)}$$

set-valued

that has a dual role with respect to the hom functor of X.

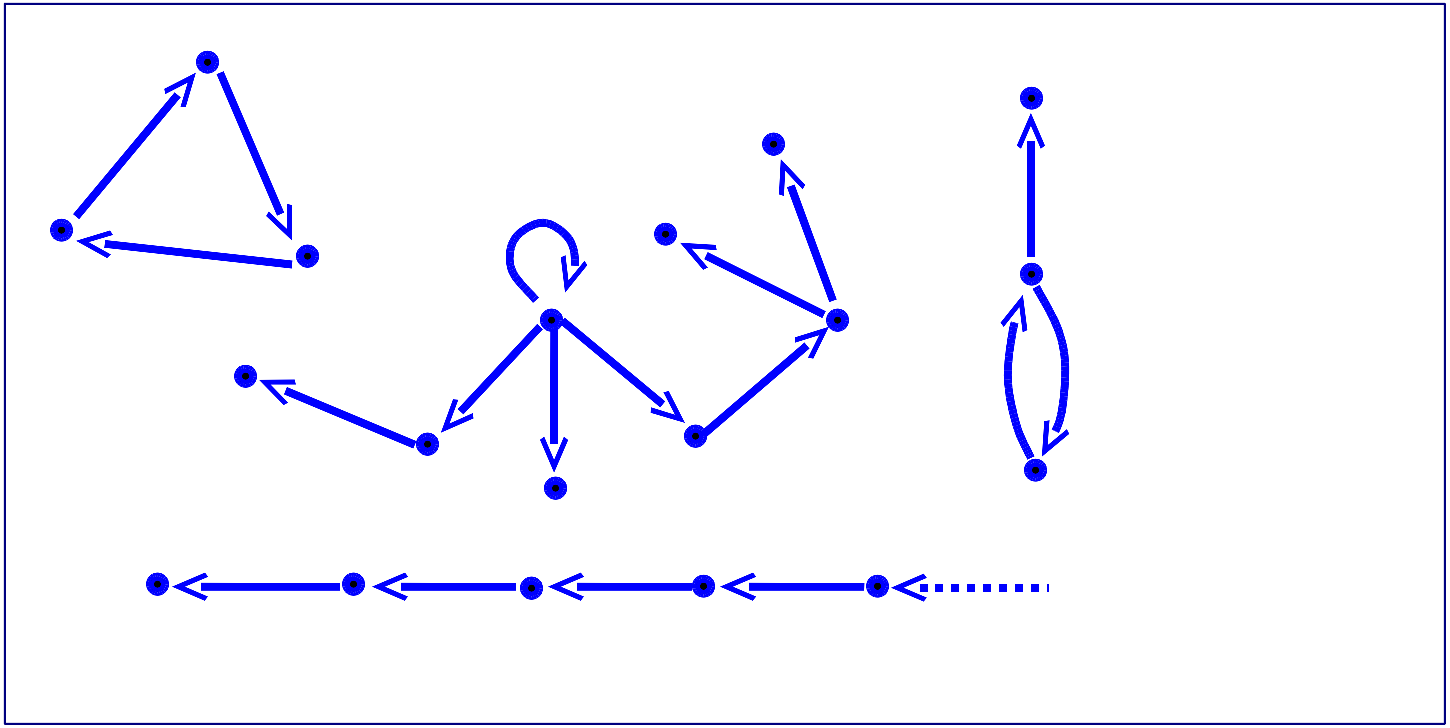
Note e.g. that if X is cartesian closed, then

$$\mathrm{hom}(x, y) = \Gamma_*(y^x)$$



Among the graphs there are the evolutive sets, or endomappings, or discrete dynamic systems:

exactly an arrow out from each node



… and the “anti-evolutive” sets.

So we have the inclusions

$$\boxed{i : \overrightarrow{\mathcal{G}} \hookrightarrow \mathcal{G}} \qquad \boxed{j : \overleftarrow{\mathcal{G}} \hookrightarrow \mathcal{G}}$$

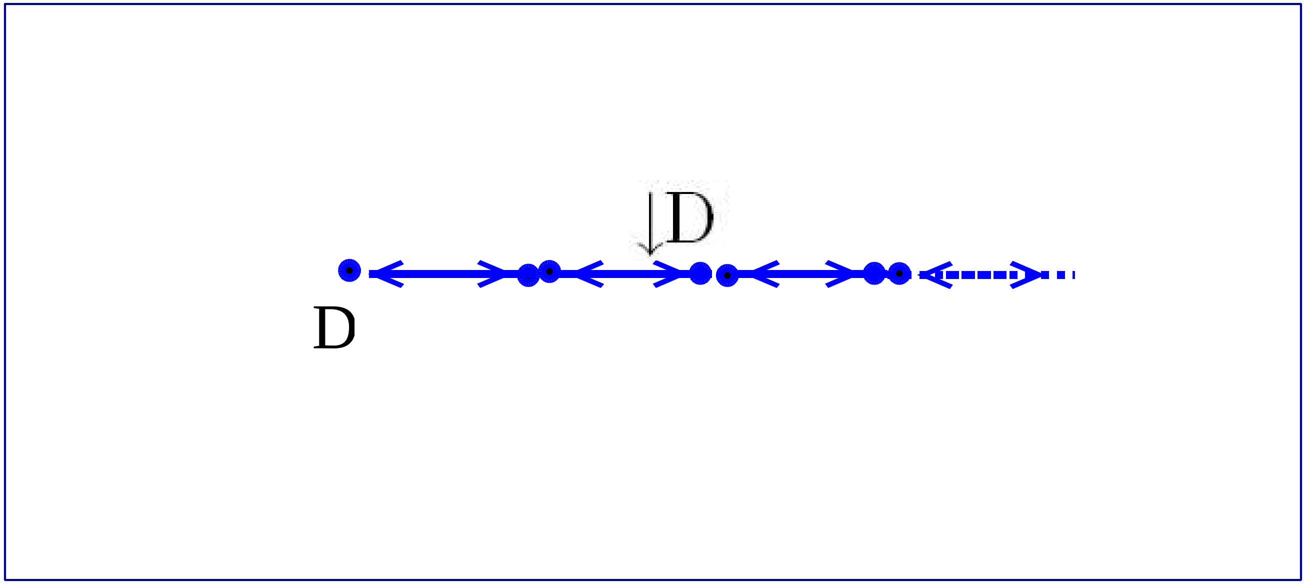
of endomappings in all the graphs, as evolutive or anti-evolutive sets respectively.

Such inclusions have both left and right adjoints

$$\boxed{\uparrow(-) \dashv i \dashv (-)\uparrow} \qquad \boxed{\downarrow(-) \dashv j \dashv (-)\downarrow}$$

What about the formulas?

The role of the points of the plane is now played by the dot graph D, whose reflections are the chain and the anti-chain:



Now, the formulas for the coreflection and the reflection of the parts of a poset in upper parts, have the following correspective for those of graphs in endomappings:

$$\frac{x \in P\uparrow}{\uparrow x \subseteq P} \quad \text{iff}$$

coreflection

$$\frac{\text{elements of } G\uparrow}{\hom(\uparrow D, G)} \quad \text{bijection}$$

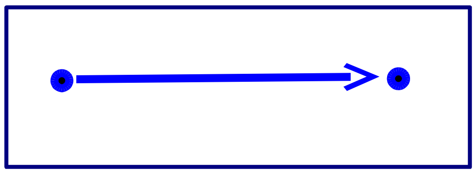
$$\frac{x \in \uparrow P}{\downarrow x \pitchfork P} \quad \text{iff}$$

reflection

$$\frac{\text{elements of } \uparrow G}{\mathrm{ten}(\downarrow D, G)} \quad \text{bijection}$$

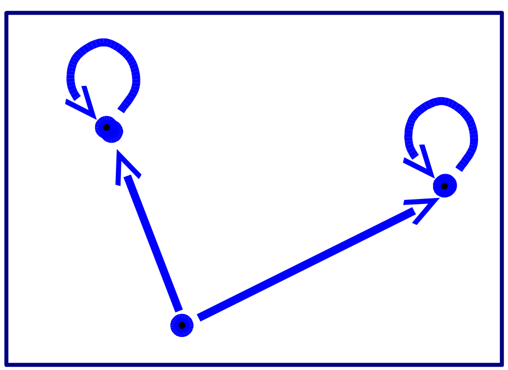
the actions being given by the shift of the chain and the anti-chain respectively.

Let’s see what these formulas give in two typical cases of non functional graphs.



A

no arrows out from the node on the right



B

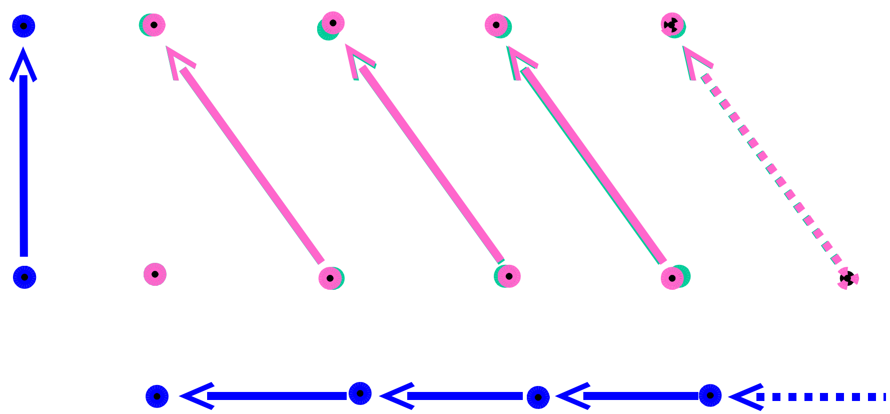
two arrows out from the lower node

$\boxed{\hom(\uparrow D, A) = \emptyset}$

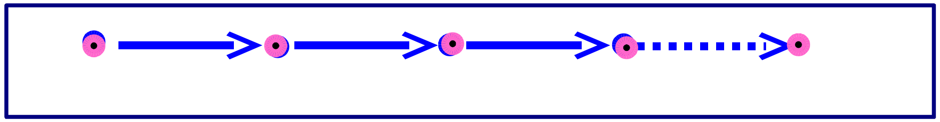
The graph A has no chains, so its coreflection is the void endomap.

domains deleted

Its reflection is given by $\boxed{\mathrm{ten}(\downarrow D, A)}$ that is by the components of the product with the anti-chain:

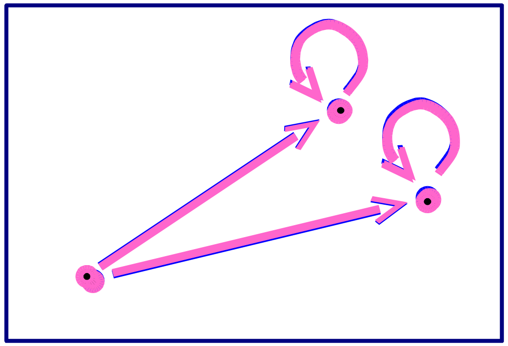


So, the reflection of A in endomappings is the chain



codomains added

The graph B

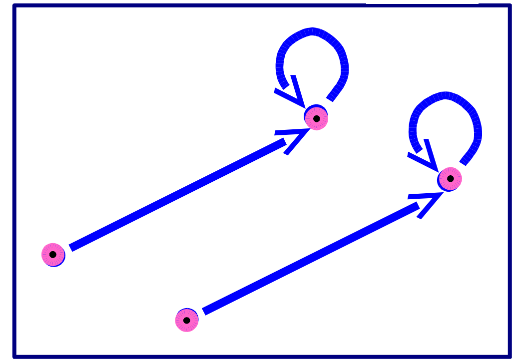


there are four chains

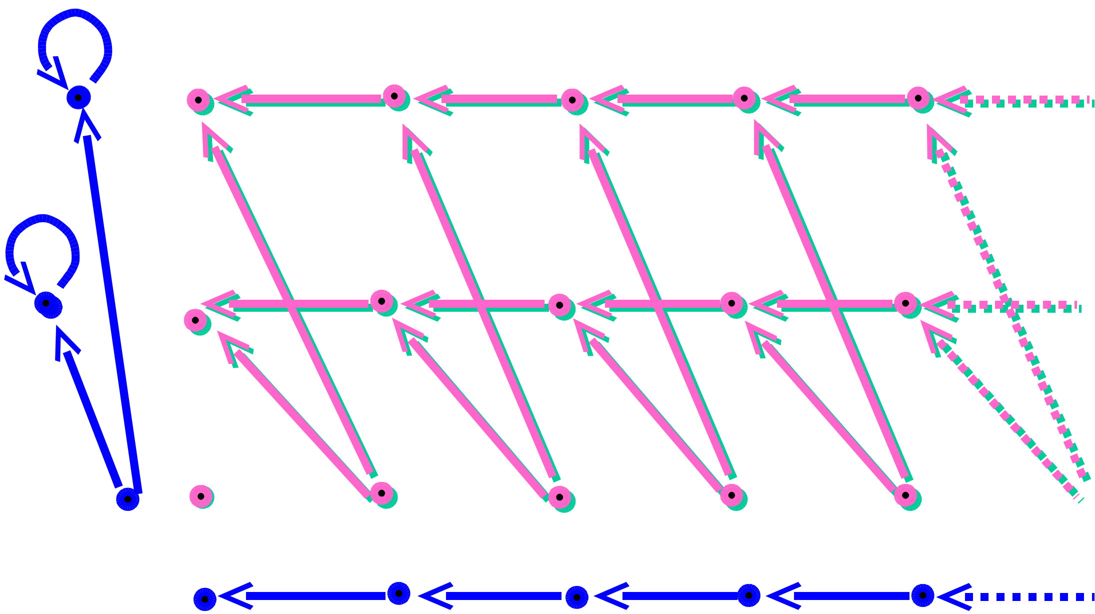
$\uparrow\! D \to B$ in

$\hom(\uparrow\! D, B)$

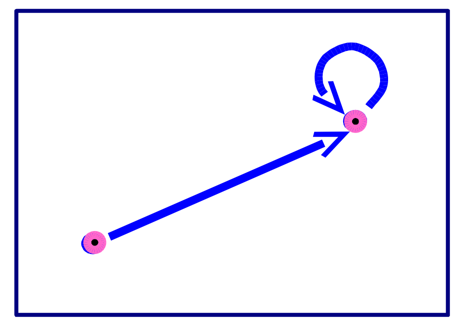
coreflection $B\!\uparrow$



domains separated



reflection $\uparrow\! B$



codomains collapsed

Let's turn to our main concern:

categories over a base category X.

We have the inclusions of full subcategories

$$\boxed{i : \overleftarrow{X} \hookrightarrow \mathbf{Cat}/X} \qquad \boxed{j : \overrightarrow{X} \hookrightarrow \mathbf{Cat}/X}$$

of discrete fibrations and discrete op-fibrations in all categories over X.

Also in this case we have functors

$$\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : \mathbf{Cat}/X \to \mathbf{Set}$$

Given a category $p : \mathrm{P} \to \mathrm{X}$ over X, its

points are the sections of p, while its components

are those of the "total category" P.

So we have the ten functor

$$\mathrm{ten} = \Gamma_!(- \times -) : \mathbf{Cat}/X \times \mathbf{Cat}/X \to \mathbf{Set}$$

It is well known that there are equivalences of categories

$$\boxed{\overleftarrow{X} \simeq \mathbf{Set}^{X^{\mathrm{op}}}} \qquad \boxed{\overrightarrow{X} \simeq \mathbf{Set}^{X}}$$

One can easily prove that, modulo these equivalences, the ten functor extends the usual tensor product of set functors (hence the name):

$$\begin{tikzcd}
\boxed{\otimes : \mathbf{Set}^{X^{\mathrm{op}}} \times \mathbf{Set}^{X} \to \mathbf{Set}} \arrow[d] \arrow[d, equal] \\
\mathrm{ten} : \mathbf{Cat}/X \times \mathbf{Cat}/X \to \mathbf{Set}
\end{tikzcd}$$

We are looking for left and right adjoints

$$\boxed{\downarrow(-) \dashv i \dashv (-)\downarrow} \qquad \boxed{\uparrow(-) \dashv j \dashv (-)\uparrow}$$

of the inclusion of df's and dof's.

The role of the points of the plane in the poset case is now played by the objects of X:
any object x of X, considered as a category over X

$$\boxed{\mathrm{x} : 1 \to \mathrm{X}}$$

has reflections $\boxed{\downarrow\mathrm{x} = \mathrm{X}/\mathrm{x}}$ and $\boxed{\uparrow\mathrm{x} = \mathrm{x}/\mathrm{X}}$

the categories of objects over and under x, which under the above equivelence correspond to the representable functors

$$\boxed{\mathrm{X}(-, \mathrm{x})} \qquad \boxed{\mathrm{X}(\mathrm{x}, -)}$$

Proof: the Yoneda Lemma.

$$\boxed{\dfrac{x \longrightarrow A}{\downarrow x \longrightarrow A}}$$

Both represent the objects over x of the df A, that is the elements of Ax .

Now, the formulas for the coreflection and the reflection of the parts of a poset in upper parts, have the following correspective for those of categories over a base in discrete opfibrations.

$$\frac{x \in P\!\uparrow}{\uparrow\! x \subseteq P} \quad \text{iff} \qquad \boxed{\text{coreflection}} \qquad \frac{(P\!\uparrow)x}{\hom(\uparrow\! x, P)} \quad \text{bijection}$$

$$\frac{x \in \uparrow\! P}{\downarrow\! x \pitchfork P} \quad \text{iff} \qquad \boxed{\text{reflection}} \qquad \frac{(\uparrow\! P)x}{\mathrm{ten}(\downarrow\! x, P)} \quad \text{bijection}$$

which give the elements of the discrete fiber over x.

Observing that

$$\boxed{X/x \ \times \ P \ = \ P/x}$$

we find

$$\boxed{(\uparrow P)x = \mathrm{ten}(\downarrow x, P) = \Gamma_!(X/x \ \times \ P) = \Gamma_!(P/x)}$$

which is the well known formula (Paré, Lawvere, …), expressing the reflection in dof's with the components of P/x.

The proof that these formulas give indeed the desired right and left adjoints are staightforward and in a sense “dual”. As far as I know, there are no published works about coreflexivity in df’s.

Another question is why the formulas have that form.

Perhaps surprisingly, the almost obvious proofs for the two-valued context (posets) fairly generalize to the present set-valued context, not only for the coreflection but also for the reflection.

So for a while we get back to posets.

As is well-known, right adjoints are easily analized with figures, and so their form is often readly determined.

Remarkably, there is an analogous deduction for the reflection formula:

$$
\begin{array}{c}
x \in P{\uparrow} \\ \hline
x \subseteq P{\uparrow} \\ \hline
{\uparrow}x \subseteq P{\uparrow} \\ \hline
{\uparrow}x \subseteq P
\end{array}
\qquad\qquad
\begin{array}{cl}
x \in {\uparrow}P & \\ \hline
x \pitchfork {\uparrow}P & * \\ \hline
{\downarrow}x \pitchfork {\uparrow}P & ** \\ \hline
{\downarrow}x \pitchfork P &
\end{array}
$$

$$
\frac{P \pitchfork D}{{\downarrow}P \pitchfork D}\,* \qquad \frac{A \pitchfork P}{A \pitchfork {\uparrow}P}\,**
$$

for any lower set A

and any upper set D

Though these are easily checked, we need to be explicit to have a proof valid in the set-valued context as well:

$$P \pitchfork D \vdash \mathrm{false}$$

$$\Gamma_!(P \cap D) \vdash \mathrm{false}$$

$$P \cap D \subseteq \Gamma^*(\mathrm{false})$$

$$P \cap D \subseteq \emptyset$$

$$P \subseteq D \Rightarrow \emptyset$$

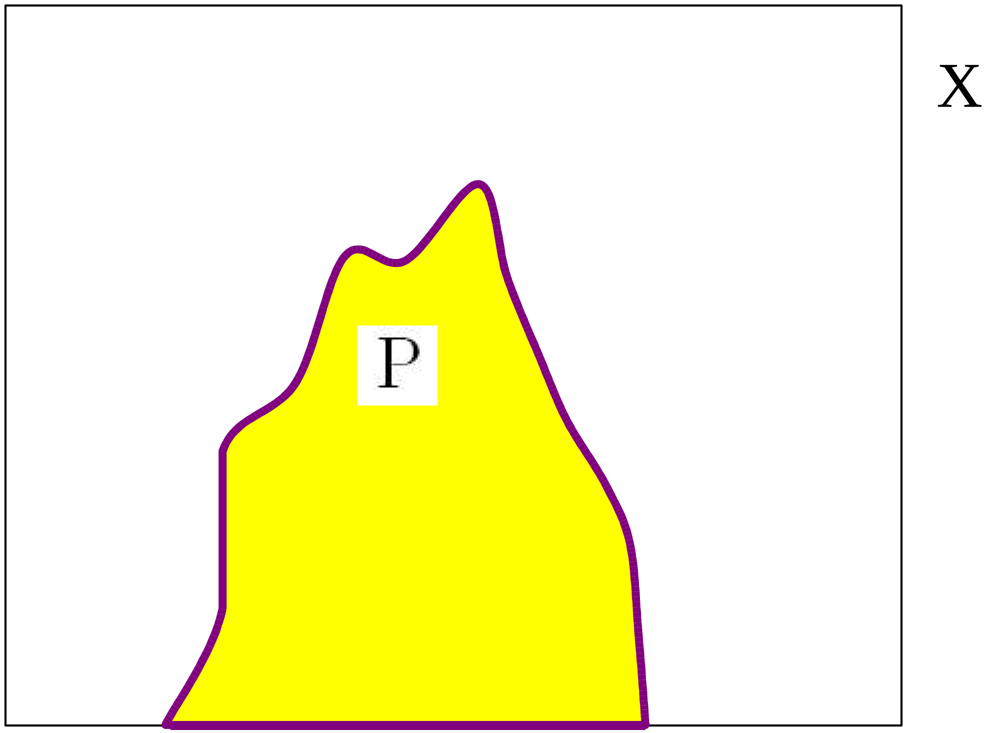
$$P \subseteq \neg D$$

$${\downarrow}P \subseteq \neg D \qquad *$$

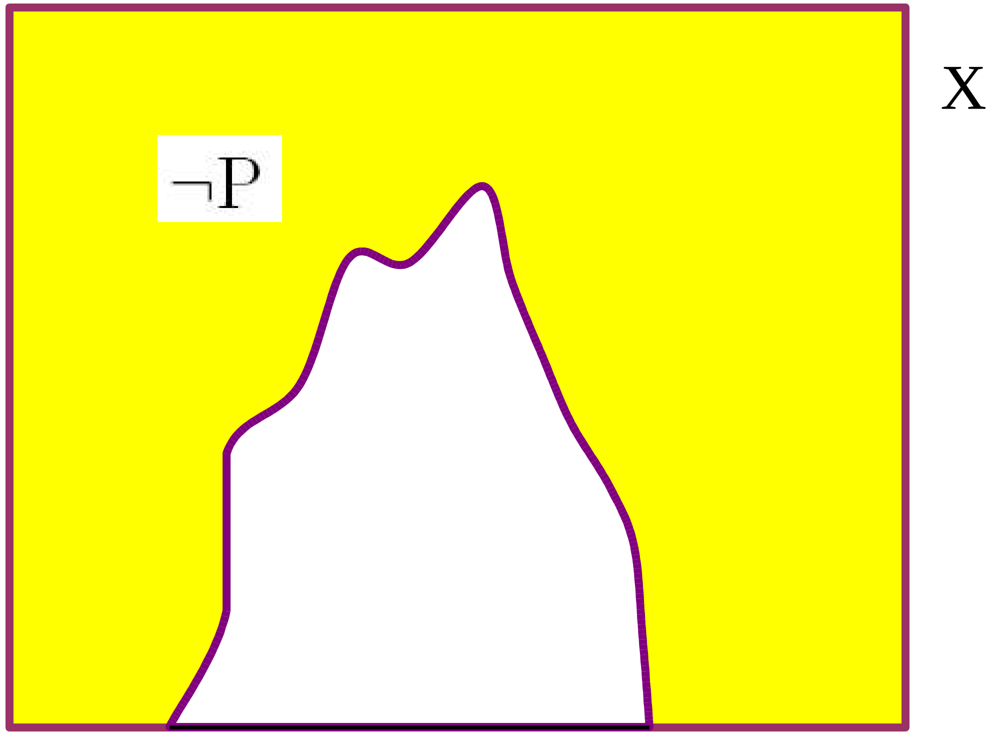
$${\downarrow}P \cap D \subseteq \emptyset$$

$${\downarrow}P \pitchfork D \vdash \mathrm{false}$$

We only have to justify the marked step:



The classical complement (in $\mathcal{P}X$) of a lower set



The classical complement (in $\mathcal{P}X$) of a lower set

is an upper set (and vice versa).

The derivation of the coreflection formula in the two-valued context straightforwardly extends to the set-valued context (e.g. categories over a base and df’s, but also graphs and evolutive sets).

$\boxed{\text{coreflection}}$

$$\begin{array}{c}
x \in P\!\uparrow \\ \hline
x \subseteq P\!\uparrow \\ \hline
\uparrow\! x \subseteq P\!\uparrow \\ \hline
\uparrow\! x \subseteq P
\end{array}
\qquad\qquad
\begin{array}{c}
(P\!\uparrow)x \\ \hline
\hom(x, P\!\uparrow) \\ \hline
\hom(\uparrow\! x, P\!\uparrow) \\ \hline
\hom(\uparrow\! x, P)
\end{array}$$

More interestingly, also the derivation of the reflection formula extends to the set-valued context:

$\boxed{\text{reflection}}$

$$\begin{array}{cl} x \in \uparrow P & \\ \cline{1-1} x \pitchfork \uparrow P & * \\ \cline{1-1} \downarrow x \pitchfork \uparrow P & ** \\ \cline{1-1} \downarrow x \pitchfork P & \end{array} \qquad\qquad \begin{array}{cl} (\uparrow P)x & \\ \cline{1-1} \mathrm{ten}(x, \uparrow P) & * \\ \cline{1-1} \mathrm{ten}(\downarrow x, \uparrow P) & ** \\ \cline{1-1} \mathrm{ten}(\downarrow x, P) & \end{array}$$

and the proof of the two marked steps is the “same” as the two-vaued one:

$$
\begin{array}{rcr}
\mathrm{P} \pitchfork \mathrm{D} \vdash \mathrm{false} & \qquad & \mathrm{ten}(\mathrm{P}, \mathrm{D}) \to \mathrm{S} \\
\Gamma_!(\mathrm{P} \cap \mathrm{D}) \vdash \mathrm{false} & & \Gamma_!(\mathrm{P} \times \mathrm{D}) \to \mathrm{S} \\
\mathrm{P} \cap \mathrm{D} \subseteq \Gamma^*(\mathrm{false}) & & \mathrm{P} \times \mathrm{D} \to \Gamma^*\mathrm{S} \\
\mathrm{P} \cap \mathrm{D} \subseteq \emptyset & & \\
\mathrm{P} \subseteq \mathrm{D} \Rightarrow \emptyset & & \mathrm{P} \to (\Gamma^*\mathrm{S})^{\mathrm{D}} \\
\mathrm{P} \subseteq \neg\mathrm{D} & & * \quad \mathrm{P} \to (\neg\mathrm{D})\mathrm{S} \\
{\downarrow}\mathrm{P} \subseteq \neg\mathrm{D} & & ** \quad {\downarrow}\mathrm{P} \to (\neg\mathrm{D})\mathrm{S} \\
{\downarrow}\mathrm{P} \cap \mathrm{D} \subseteq \emptyset & & {\downarrow}\mathrm{P} \times \mathrm{D} \to \Gamma^*\mathrm{S} \\
{\downarrow}\mathrm{P} \pitchfork \mathrm{D} \vdash \mathrm{false} & & \mathrm{ten}({\downarrow}\mathrm{P}, \mathrm{D}) \to \mathrm{S}
\end{array}
$$

The functor $\boxed{\neg \mathrm{D} = (\Gamma^* -)^{\mathrm{D}} : \mathbf{Set} \to \mathbf{Cat}/\mathrm{X}}$

which is right adjoint to $\mathrm{ten}(\mathrm{D}, -) : \mathbf{Cat}/\mathrm{X} \to \mathbf{Set}$

$$\boxed{\mathrm{ten}(\mathrm{D}, -) \dashv \neg \mathrm{D}}$$

deserves to be called the negation or complement of D.

If D is a dof, its complement is valued in df's,
and conversely.

It is classical, that is the strong contraposition law holds:

$$\boxed{\frac{\neg \mathrm{A} \to \neg \mathrm{B}}{\mathrm{B} \to \mathrm{A}}}$$

A and B both df's
or both dof's

The meets operator allows a natural definition of atom in any (bounded) poset X:

$$\boxed{\dfrac{x \leq y}{x \pitchfork y}}$$

for any y in X

That is, x is an element “so small” that it is included in any element that it meets.

But also“big enough” to meet any element in which it is included (the bottom is included in any element, but doesn’t meet them).

In the set-valued context, given a category X with components

$$\Gamma_! \dashv \Gamma^* \dashv \Gamma_* : X \to \mathbf{Set}$$

an object x is an atom iff

$$\boxed{\frac{\mathrm{hom}(x,\ y)}{\mathrm{ten}(x,\ y)}}$$

natural in y

E.g., in the category of graphs, the dot graph is an atom.

Indeed, for any other graph y, both sets represent the nodes of y (multiplying by the dot graph has the effect of deleting arrows).

In the case of categories over a base

$\boxed{\mathrm{hom}(x, P)}$ and $\boxed{\mathrm{ten}(x, P)}$

are the set of objects and of components of the fiber Px over x, respectively.

But these coincide if the fiber is discrete.

So the objects x of the base category X are atoms in the weaker sense that the bijection

$$\frac{\mathrm{hom}(x, P)}{\mathrm{ten}(x, P)}$$

holds for discrete fibrations (or opfibrations) P.

Are there other atoms?

Yes: any idempotent arrow in X is an atom!

Indeed, given any idempotent arrow in X

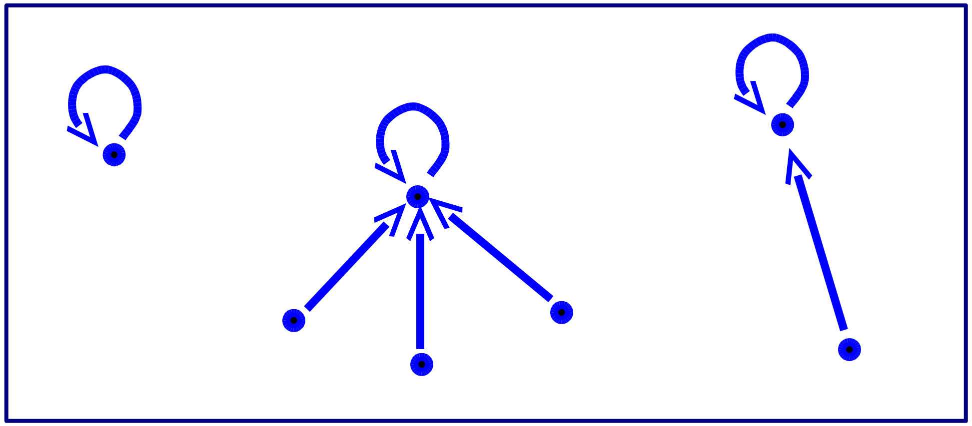
$e : x \to x$ considered as a category over X

$\boxed{e : \mathbf{e} \to X}$ and any df A over X

$$\hom(e, A) \quad = \quad \text{fixed points of the endomap } Ae$$

$$\mathrm{ten}(e, A) \quad = \quad \text{components of the endomap } Ae$$

and these coincide for idempotent mappings:



What are the reflections of an idempotent $e : y \to y$?

$(\downarrow e)x$ is the set of arrows $f : x \to y$

such that $\boxed{e \circ f = f}$

If e splits, $\downarrow e$ is isomorphic to the representable $\downarrow y$

In general, $\downarrow e$ is a retract of the representable $\downarrow y$ which splits the idempotent

$e : \downarrow y \to \downarrow y$ in $\mathbf{Set}^{X^{op}}$

So the reflections of idempotents generate the Chaucy completion of X.

Given two idempotents $\boxed{e : x \longrightarrow x}$ and $\boxed{e' : y \longrightarrow y}$

$$\hom(\downarrow e, \downarrow e')$$

$$\hom(e, \downarrow e')$$

---

elements of $(\downarrow e')x$ fixed by $(\downarrow e')e$

$f : x \longrightarrow y$ such that $f \circ e = f$ and $e' \circ f = f$

That is, the Cauchy completion of X is equivalent to the Karoubi envelope of X.

All this, and much more, can be found in the preprint:

Bipolar spaces

available in arXiv

where we try capture the scope of the above formalism.