$(X, *)$ pointed topological space

ΩX the space of loops at $*$

$m : \Omega X \times \Omega X \to \Omega X$ only homotopy associative

$e : 1 \to \Omega X$ only a homotopy unit

ΩX homotopy monoid

$\mathcal{T}$ algebraic theory of monoids

homotopy monoid $A : \mathcal{T} \to \mathbf{Top}$

$$A(X_1 \times X_1) \to A(X_1) \times A(X_1)$$

and

$$A(X_0) \to 1$$

homotopy equivalences

$$m_A : A(X_1) \times A(X_1) \to A(X_1 \times X_1) \to A(X_1)$$

morphisms preserve operations up to homotopy

$\mathcal{T}$ algebraic theory

homotopy $\mathcal{T}$*-algebra* $A : \mathcal{T} \to \mathbf{SSet}$

$$A(X_1 \times \cdots \times X_n) \to A(X_1) \times \cdots \times A(X_n)$$

homotopy equivalences (weak equivalences)

no difference if $A : \mathcal{T} \to \mathbf{S}$

Badzioch 2002, Bergner 2005

simplicial category $\mathcal{K}$ = category enriched over $\mathbf{SSet}$

$\mathcal{D}$ category, $D : \mathcal{D} \to \mathcal{K}$

$holim_s D$ limit of D weighted by

$$B(\mathcal{D} \downarrow -) : \mathcal{D} \to \mathbf{SSet}$$

$B(\mathcal{X})$ the nerve of $\mathcal{X}$

$\mathbf{SSet}^{\mathcal{T}}$ simplicial model category

weak equivalences and fibrations are pointwise

$A : \mathcal{T} \to \mathbf{SSet}$ fibrant iff $A(X) \in \mathbf{S}$ for each $X \in \mathcal{T}$

$$\mathbf{HAlg}(\mathcal{T}) \subseteq \mathbf{SSet}^{\mathcal{T}}$$

consists of simplicial functors

$$A : \mathcal{T} \to \mathbf{S}$$

which are cofibrant and

$$A(X_1 \times \cdots \times X_n) \to A(X_1) \times \cdots \times A(X_n)$$

are homotopy equivalences

$$holim\, D = R_c(holim_s D)$$

$$\mathbf{HAlg}(\mathcal{T}) \subseteq \mathbf{SSet}^{\mathcal{T}}$$

closed under homotopy limits

$$\mathrm{hocolim} D = R_f(\mathrm{hocolim}_s D)$$

closed under homotopy sifted colimits

$\mathcal{D}$ homotopy sifted

homotopy colimits over $\mathcal{D}$ homotopy commute with finite products in $\mathbf{S}$

Theorem. *$\mathcal{D}$ homotopy sifted iff*

$$\Delta : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$$

homotopy final

iff $(A, B) \downarrow \Delta$ *aspherical.*

$$A \to X \leftarrow B$$

homotopy final $\Rightarrow$ final

aspherical $\Rightarrow$ connected

homotopy sifted $\Rightarrow$ sifted

$\mathcal{D}$ homotopy sifted iff $\mathcal{D}^{\mathrm{op}}$ totally aspherical

Grothendieck, Maltsiniotis

When $\mathbf{Set}^{\mathcal{D}}$ has the homotopy category equivalent with that of $\mathbf{Top}$?

filtered $\Rightarrow$ homotopy sifted

with finite coproducts $\Rightarrow$ homotopy sifted

coequalizers of reflexive pairs

$$f_1, f_2 : A_1 \to A_0$$

sifted but not homotopy sifted

$$(A_1, A_1) \downarrow \Delta$$

connected but not 2-connected

$\mathbf{\Delta}^{\mathrm{op}}$ homotopy sifted

$$\mathbf{HAlg}(\mathcal{T}) \subseteq \mathbf{SSet}^{\mathcal{T}}$$

closed under homotopy sifted homotopy colimits

Goal: An abstract characterization of categories "equivalent" to $\mathbf{HAlg}(\mathcal{T})$.

homotopy in simplicial categories

$$\hom_{Ho(\mathcal{K})}(K, L) = \pi_0 \hom_{\mathcal{K}}(K, L)$$

homotopy equivalences in $\mathcal{K}$

simplicial $Ho(\mathbf{SSet})$ is not $Ho(\mathbf{SSet})$

simplicial $Ho(\mathbf{S})$ is $Ho(\mathbf{S})$

$\mathcal{K}$ *fibrant* if $\hom(K, L) \in \mathbf{S}$

$\mathbf{HAlg}(\mathcal{T})$ fibrant

$F : \mathcal{K} \to \mathcal{L}$ *Dwyer-Kan equivalence*

(a) $\hom(K_1, K_2) \to \hom(FK_1, FK_2)$ homotopy (weak) equivalence

(b) each $L \in \mathcal{L}$ is homotopy equivalent to some FK

Model category structure on small simplicial categories with D-K equivalences as weak equivalences.

Fibrant objects are fibrant simplicial categories.

homotopy (co)limits in fibrant simplicial categories

$$\hom(-, holimD) \simeq holim_s \hom(-, D)$$

$$\hom(hocolimD, -) \simeq holim_s \hom(D, -)$$

coincides with the previous ones in $Int(\mathbf{SSet}^{\mathcal{T}})$

for any simplicial model category $\mathcal{M}$

$\mathcal{C}$ a small category

$$\mathbf{Pre}(\mathcal{C}) = Int(\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}})$$

prestacks on $\mathcal{C}$

Theorem. $\mathbf{Pre}(\mathcal{C})$ *is a free completion of* $\mathcal{C}$ *under homotopy colimits (among fibrant simplicial categories).*

Dugger 2001

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "Y"] \arrow[dr, "F"'] & & \mathbf{Pre}(\mathcal{C}) \arrow[dl, "F^*"] \\ & \mathcal{K} & \end{tikzcd}$$

$\mathcal{K}$ fibrant simplicial category

$K \in \mathcal{K}$ *homotopy strongly finitely presentable*

$\hom(K, -)$ preserves homotopy sifted homotopy colimits

$\mathcal{K}$ *homotopy variety*

(i) has homotopy colimits

(ii) has a set $\mathcal{A}$ of homotopy strongly finitely presentable objects such that every object is a homotopy sifted homotopy colimit of objects from $\mathcal{A}$.

$\mathbf{HAlg}(\mathcal{T})$ homotopy variety

$\mathcal{K}$ homotopy variety

$\mathcal{T}$ the dual of the full subcategory consisting of homotopy strongly finitely presentable objects

$\mathcal{T}$ *simplicial algebraic theory*

small fibrant simplicial category with finite products

any algebraic theory is a simplicial algebraic theory

Theorem. *$\mathcal{K}$ homotopy variety iff it is D-K equivalent to* $\mathbf{HAlg}(\mathcal{T})$ *for a simplicial algebraic theory* $\mathcal{T}$*.*

$\mathbf{Pre}(\mathcal{C})$ free completion under homotopy colimits for every fibrant simplicial category $\mathcal{C}$

$$\mathbf{HAlg}(\mathcal{T}) = \mathbf{HSind}(\mathcal{T}^{\mathrm{op}})$$

free completion under homotopy sifted homotopy colimits

$\mathcal{T}$ algebraic theory of monoids

put Δ_1 from $m(m \times 1)$ to $m(1 \times m)$ in $\mathcal{T}(X_1^3, X_1)$

$A : \mathcal{T} \to \mathbf{SSet}$ strict algebra

$$\Delta_1 \to \mathcal{T}(X_1^3, X_1) \to \mathbf{SSet}(A(X_1)^3, A(X_1))$$

$$\Delta_1 \times A(X_1)^3 \to A(X_1)$$

homotopy from $m_A(m_A \times 1)$ to $m_A(1 \times m_A)$

strong homotopy associativity (Stasheff)

homomorphisms are strict

Each homotopy algebra is weakly equivalent to a strict algebra (in a suitable model category structure).

Badzioch, Bergner

homotopy locally finitely presentable categories

homotopy limit theories (sketches)

homotopy accessible categories (Lurie 2003)

homotopy toposes

homotopy Giraud theorem (Lurie, Toën, Vezzosi 2002)

homotopy exactness

groupoid objects are effective

$$
\begin{tikzcd}
\cdots & X_2 \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] & X_1 \arrow[r, shift left] \arrow[r, shift right] & 1 \arrow[r] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r, "\lambda"'] & X
\end{tikzcd}
$$

characterization of loop spaces (Stasheff, Segal)