Orthogonality Logic

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Orthogonal Subcategory Problem and Orthogonality Logic

category $\mathcal{A}$, $\mathcal{H} \subseteq \mathrm{Mor}(\mathcal{A})$

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Question: When are these "*rules*" part of a sound and complete deduction system for orthogonality?

Orthogonal Subcategory Problem and Orthogonality Logic

Find a Deduction System of *RULES* such that

$h \in \left(\mathcal{H}^{\perp}\right)_{\perp} \Leftrightarrow$ h is deducible from $\mathcal{H}$ by successively applying the *RULES*

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$\mathcal{H} \models h \;:=\;$ $(A \perp \mathcal{H} \Rightarrow A \perp h)$, for all objects A

$\mathcal{H} \vdash h \;:=\;$ there is a formal proof of h from $\mathcal{H}$ by using the Deduction System

The Finitary Case: Sentences versus Morphisms

$e \equiv (u = v)$

u and v terms in X

$q_e : FX \to FX/\sim_e$

algebras satisfying

$\mathbb{E} = \{e_i, i \in I\}$, $e_i \equiv (u_i = v_i)$

algebras orthogonal to

$\mathbb{E}' = \{q_{e_i}, i \in I\}$

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Analogously for implications and regular sentences

The Finitary Case: Sentences versus Morphisms

A satisfies	A is orthogonal to
equations $\forall \mathbf{x} E(\mathbf{x})$	epimorphisms with projective domain (orthogonality=inject.)
implications $\forall \mathbf{x}(E(\mathbf{x}) \to F(\mathbf{x}))$	epimorphisms (orthogonality=inject.)
limit sentences $\forall \mathbf{x}(E(\mathbf{x}) \to \exists ! \mathbf{y} F(\mathbf{x}, \mathbf{y}))$	morphisms

$E(\mathbf{x})$ and $F(\mathbf{x})$ involving a finite number of variables and equations

finitary morphisms, i.e., with finitely presentable domain and codomain

G. Roşu, *Complete Categorical Equational Deduction* (2001):

A sound and complete deduction system for finitary epimorphisms with projective domains

Adámek, Sobral, Sousa, *Logic of implications* (2005):

A sound and complete deduction system for finitary epimorphisms

Finitary Logic

$\mathcal{A}$ a finitely presentable category

- Formulas: finitary morphisms, i.e., morphisms of $\mathcal{A}_{fp}$
- Formal proofs have only a finite number of steps

If $\mathcal{F}$ is a set of finitary morphisms admitting a left calculus of fractions (in $\mathcal{A}_{fp}$) then

$$\mathcal{F}^{\perp} \text{ is reflective in } \mathcal{A}.$$

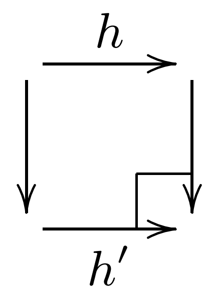
Hébert, Adámek, Rosický, More on orthogonality in l.p.c., *Cah. Topol. Géom. Différ. Catég.* 42 (2001)

sound rules

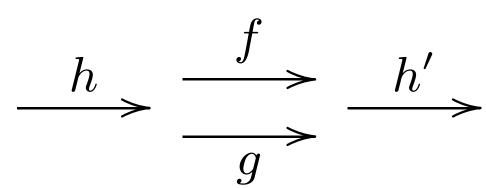
IDENTITY $\dfrac{}{\mathrm{id}_A}$

COMPOSITION $\dfrac{h_1 \quad h_2}{h_2 \cdot h_1}$

PUSHOUT $\dfrac{h}{h'}$ if



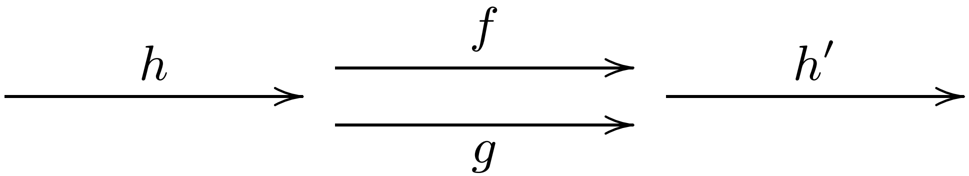
COEQUALIZER $\dfrac{h}{h'}$ if



$$f \cdot h = g \cdot h$$

$$h' = \text{coeq}(f, g)$$

Soundness of COEQUALIZER $\frac{h}{h'}$



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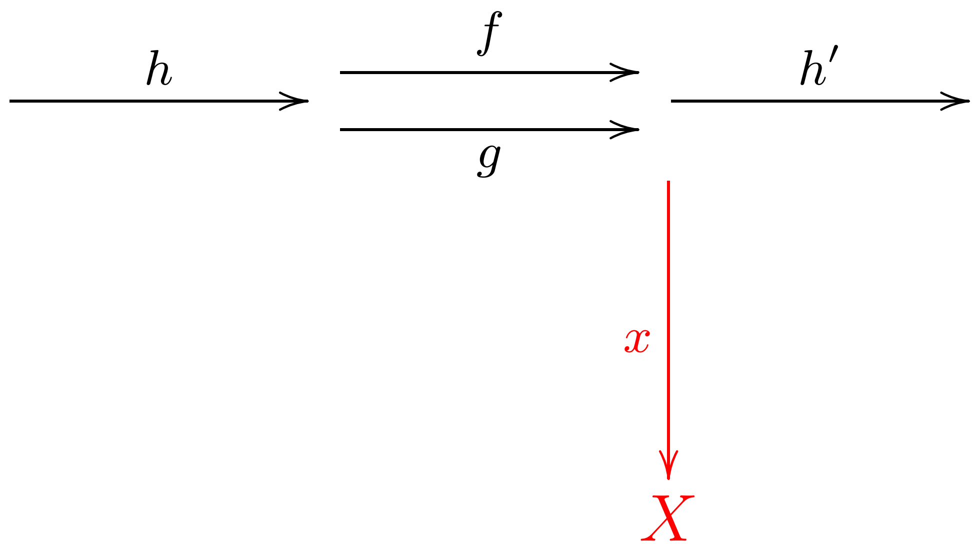

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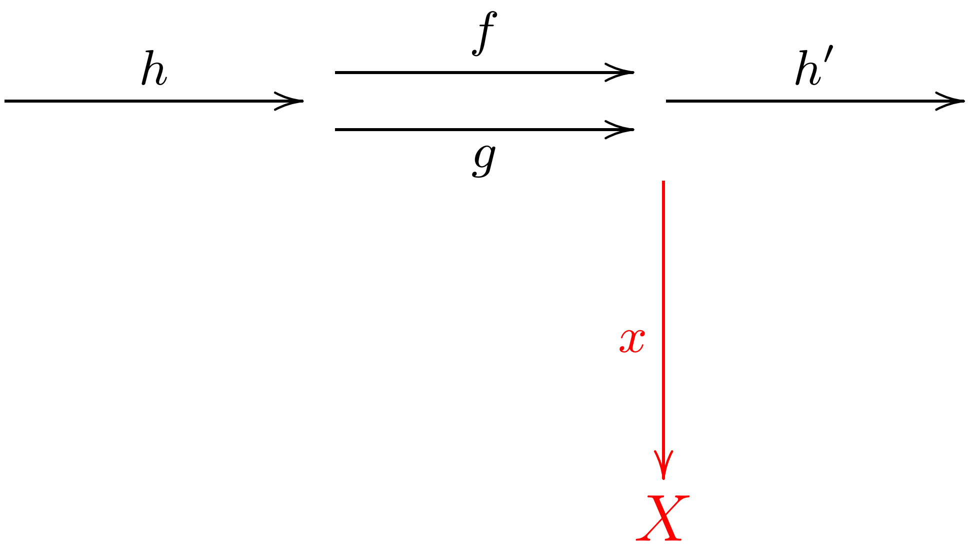

$$(xf)h = (xg)h \Rightarrow xf = xg$$

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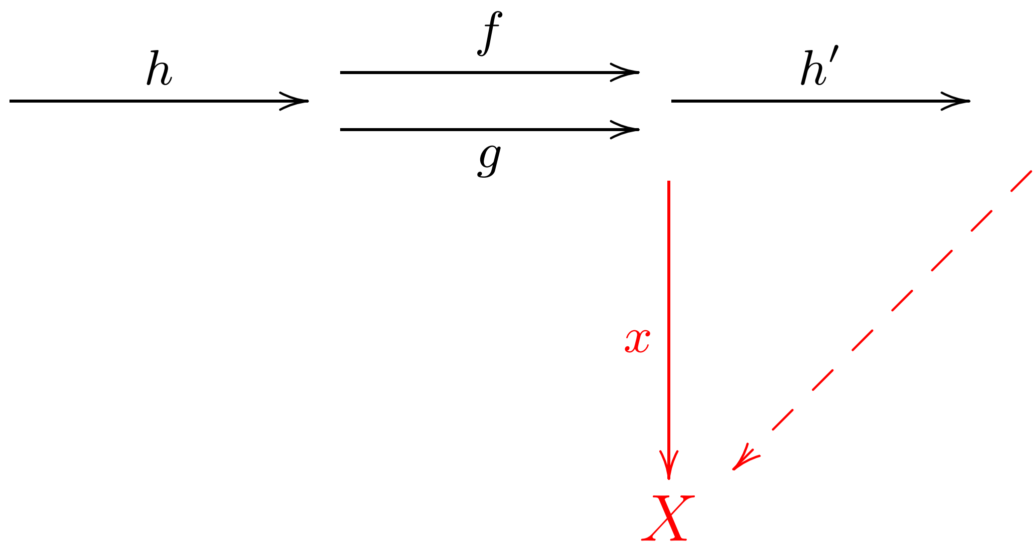

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CANCELLATION **is not sound**

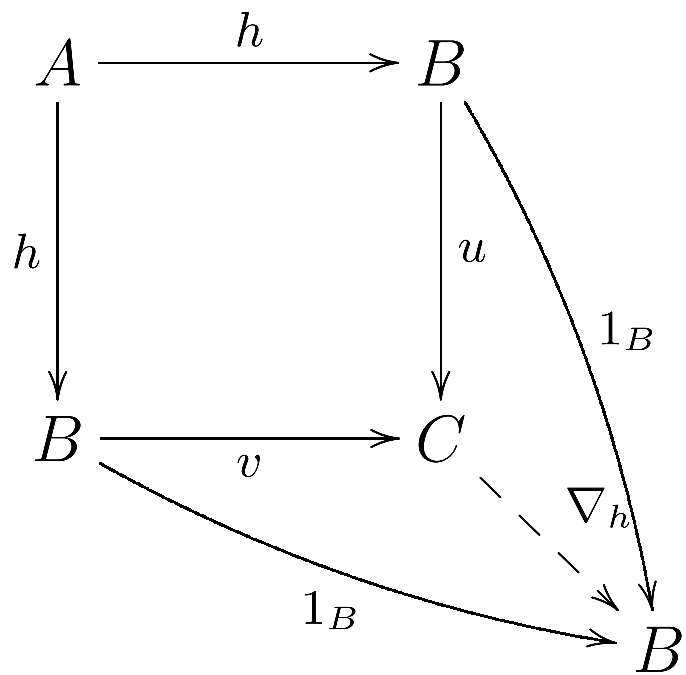
$$\{0\} \overset{f}{\hookrightarrow} \{0,1\} \overset{g}{\longrightarrow} \{0\}$$

$$g \cdot f = \mathrm{id}_{\{0\}} \not\models f$$

because $\{0,1\} \models \mathrm{id}_{\{0\}}$ but $\{0,1\} \not\models f$)

∇-CANCELLATION

$$\frac{f \cdot h \quad \nabla_h}{h}$$



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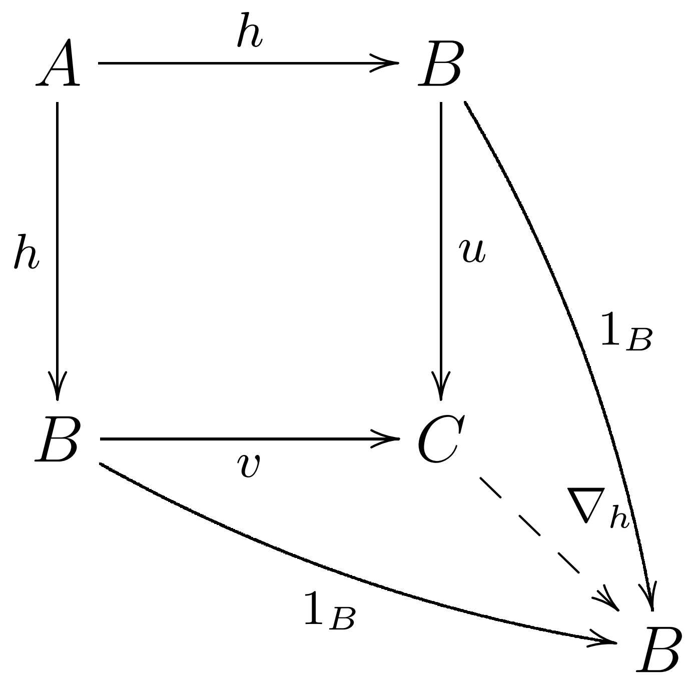

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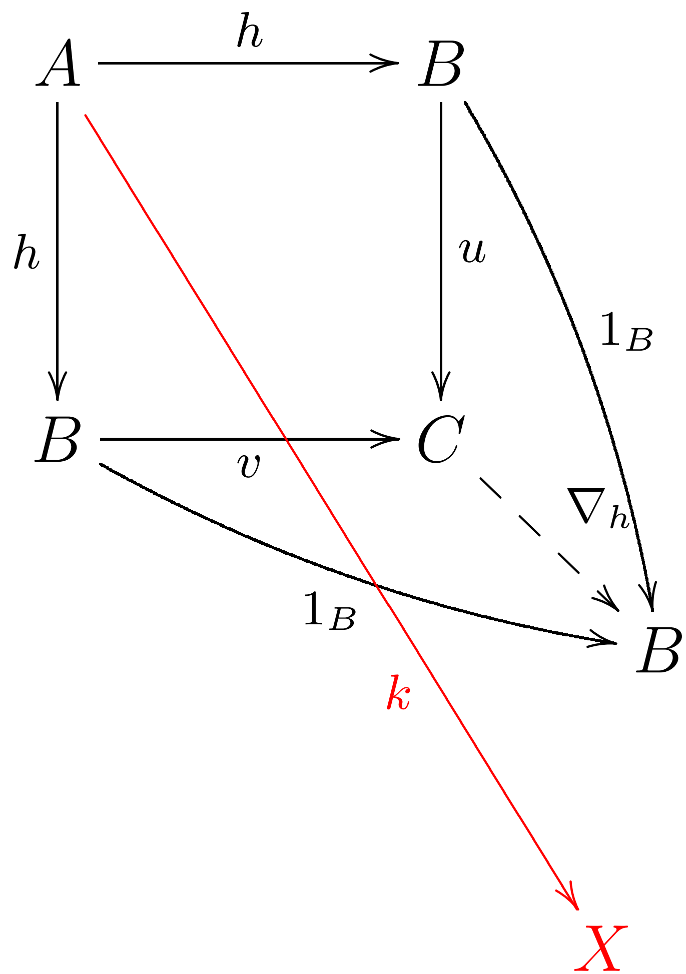

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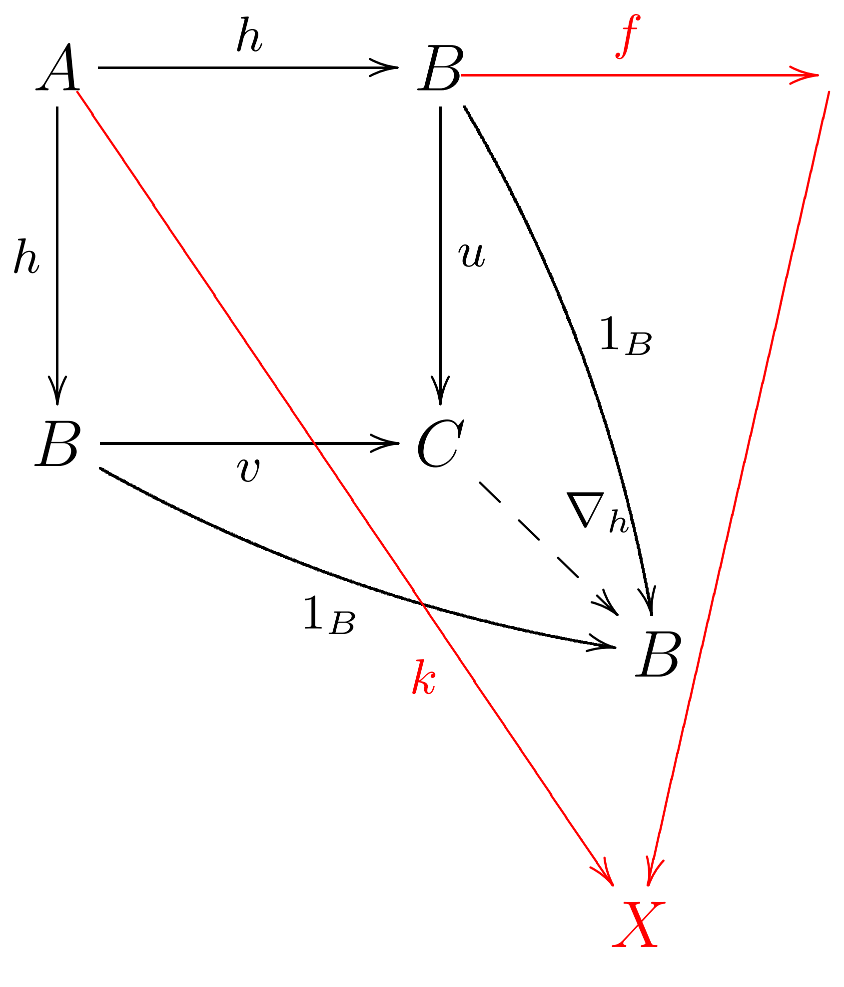

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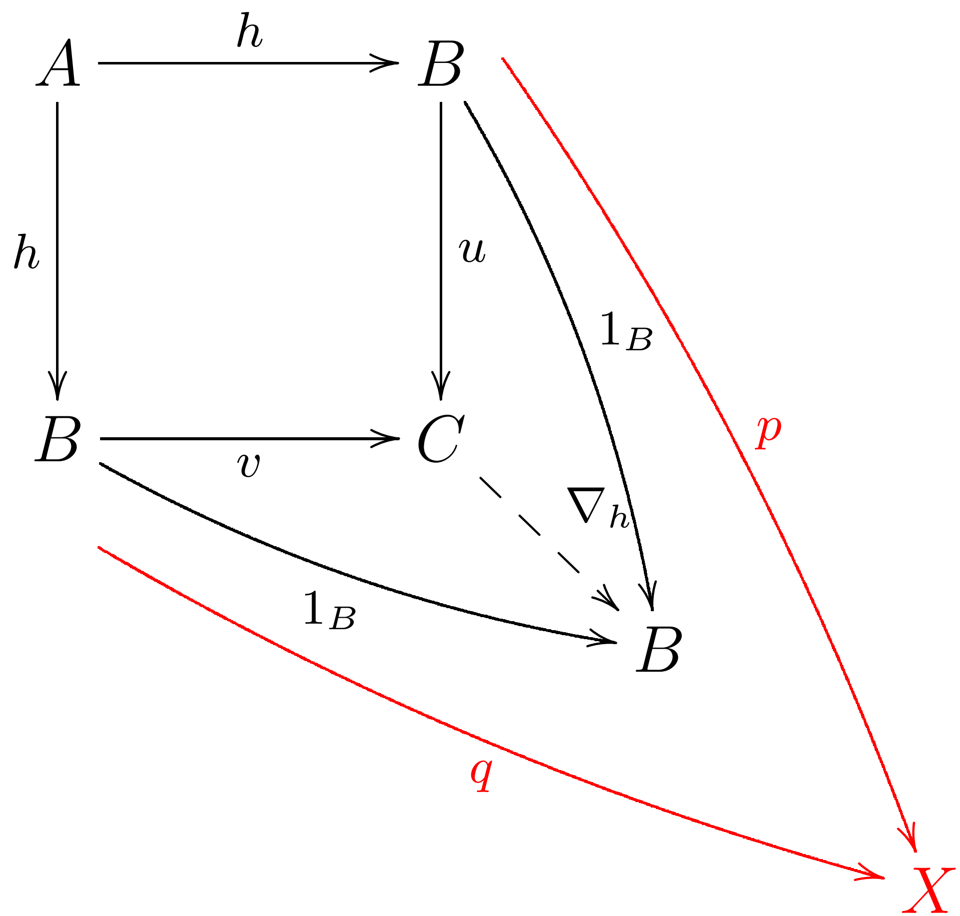

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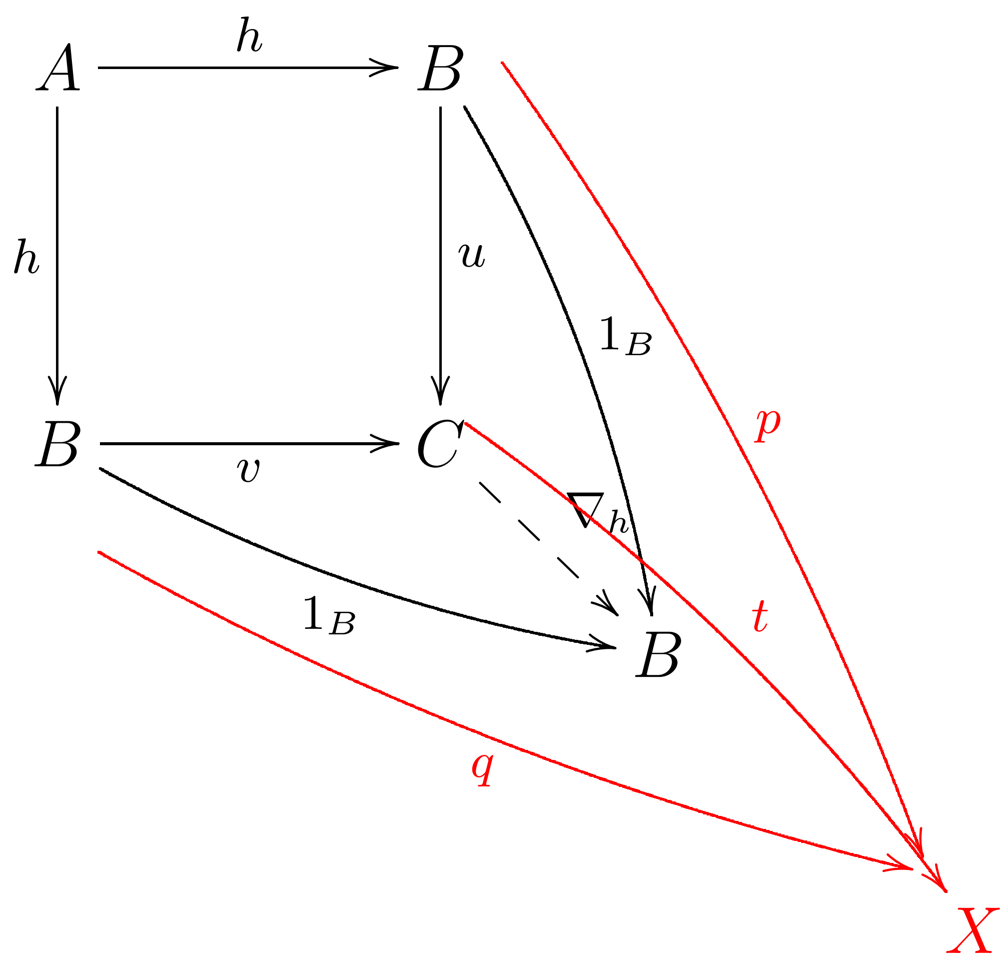

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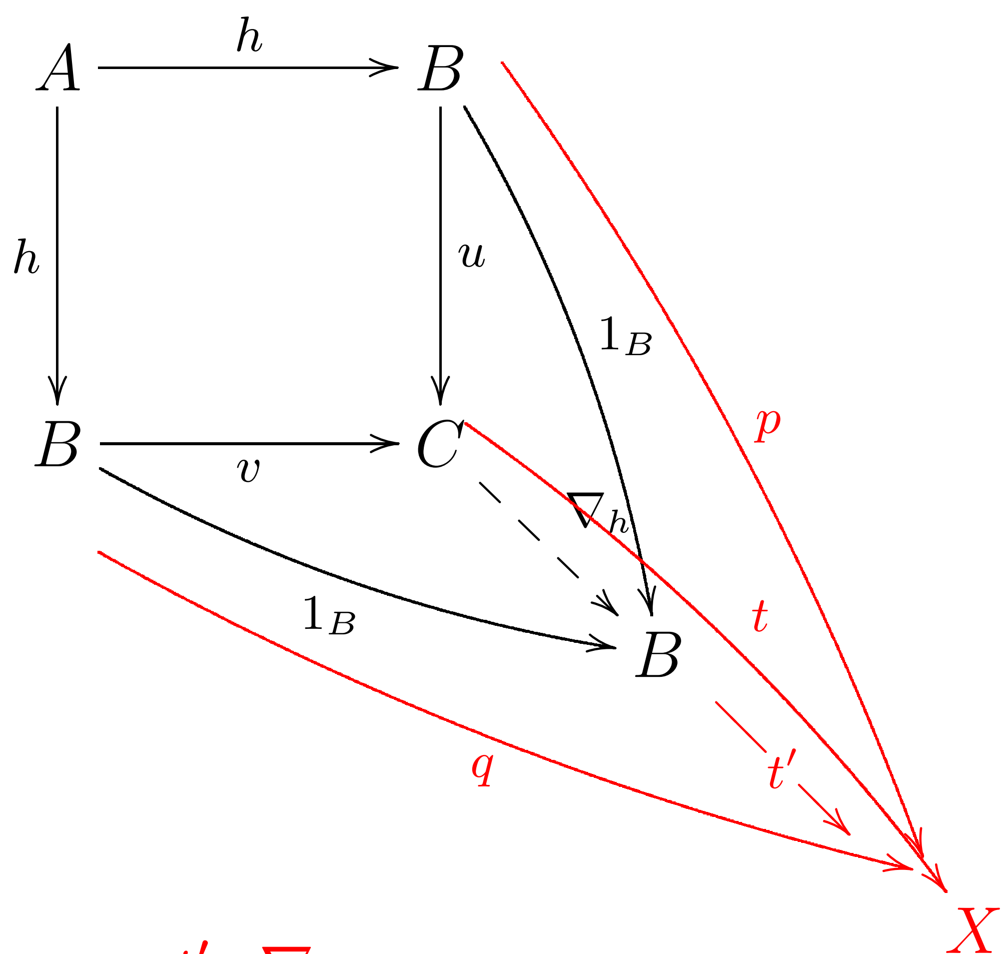


$$p = t' \cdot \nabla_h \cdot u = t' \cdot \nabla_h \cdot v = q$$

Finitary Orthogonality Deduction System

IDENTITY

$$\frac{}{\mathrm{id}_A}$$

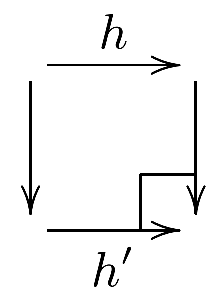
COMPOSITION

$$\frac{h_1 \quad h_2}{h_2 \cdot h_1}$$

PUSHOUT

$$\frac{h}{h'}$$

if



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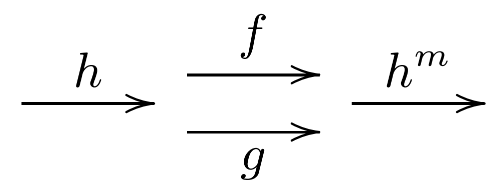

$fh = gh,\ h' = \mathsf{coeq}(f, g)$

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The Finitary Orthogonality Deduction System is sound and complete, that is,

$$\mathcal{H} \models h \text{ iff } \mathcal{H} \vdash h$$

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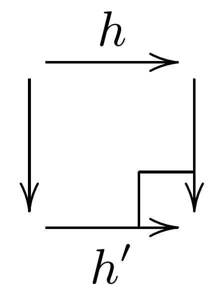

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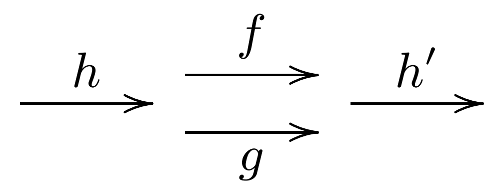

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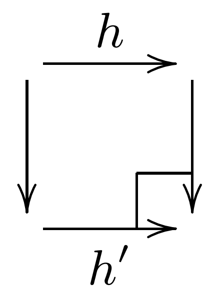

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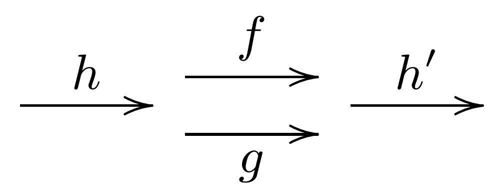

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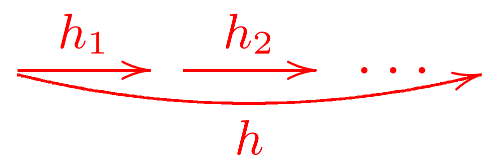
$$\frac{h_i,\ i \in \alpha}{h}$$



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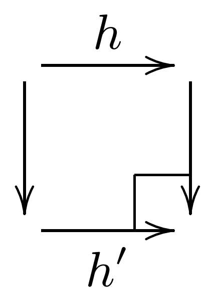

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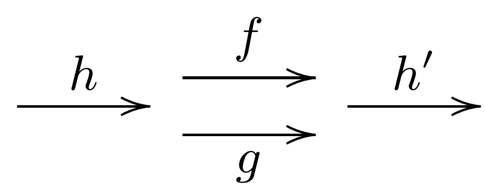

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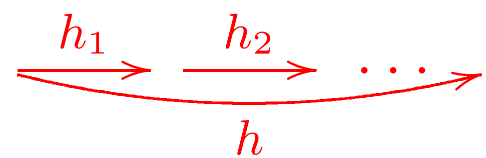

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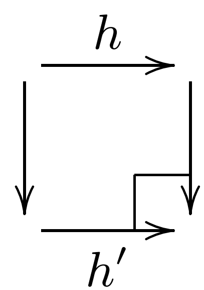

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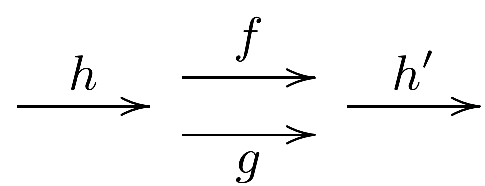

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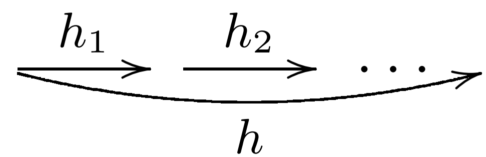

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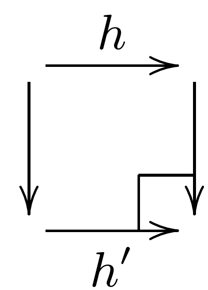

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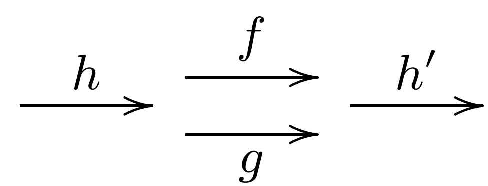

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The Orthogonality Deduction System is sound and complete.

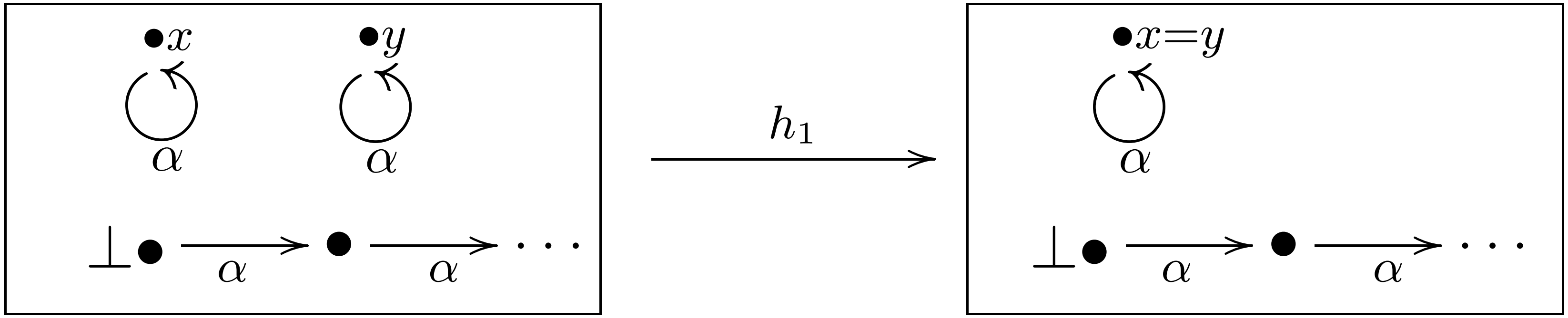
That is,

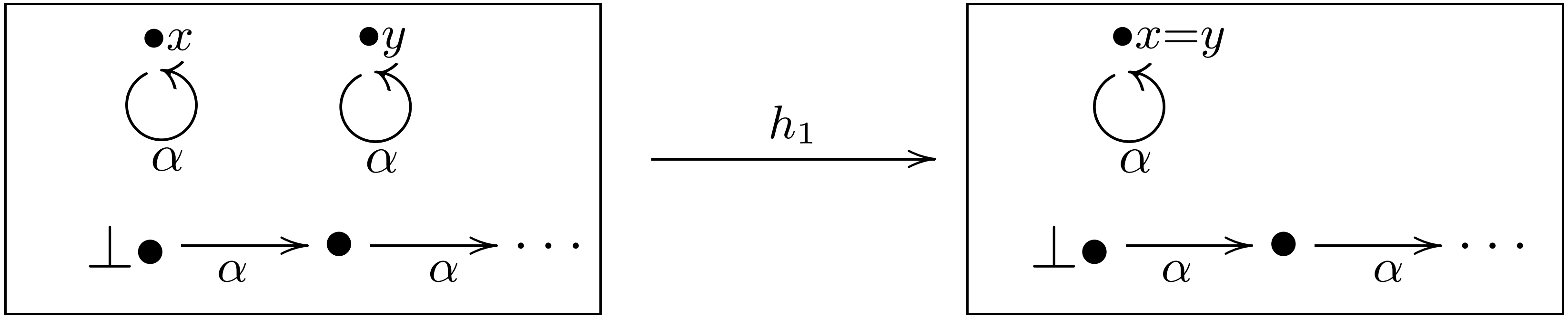
$$\mathcal{H} \models h \text{ iff } \mathcal{H} \vdash h$$

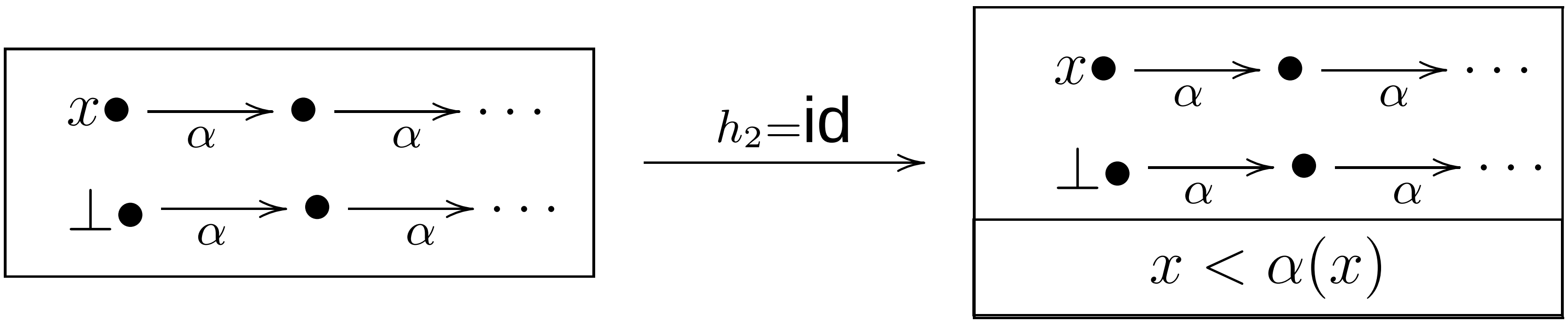
Incompleteness Example: a cocomplete category where the Orthogonality Logic is not complete

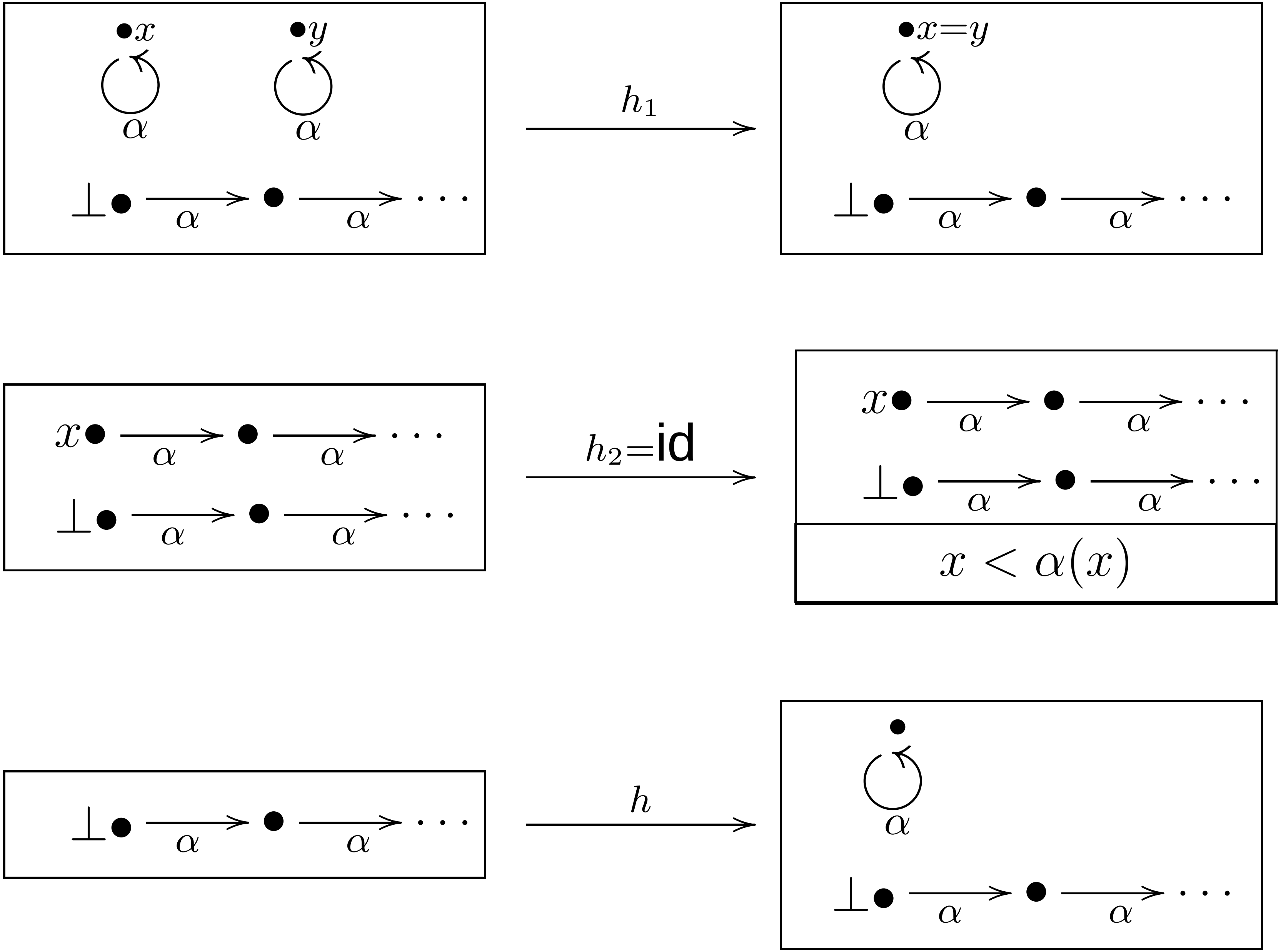
$\mathbf{CPO}_\perp(\mathbf{1})$

- Objects: $(X, \leq, \alpha)$, where $(X, \leq)$ is a CPO with a least element, and $\alpha : X \to X$
- Morphisms: continuous maps preserving the least element and the unary operation

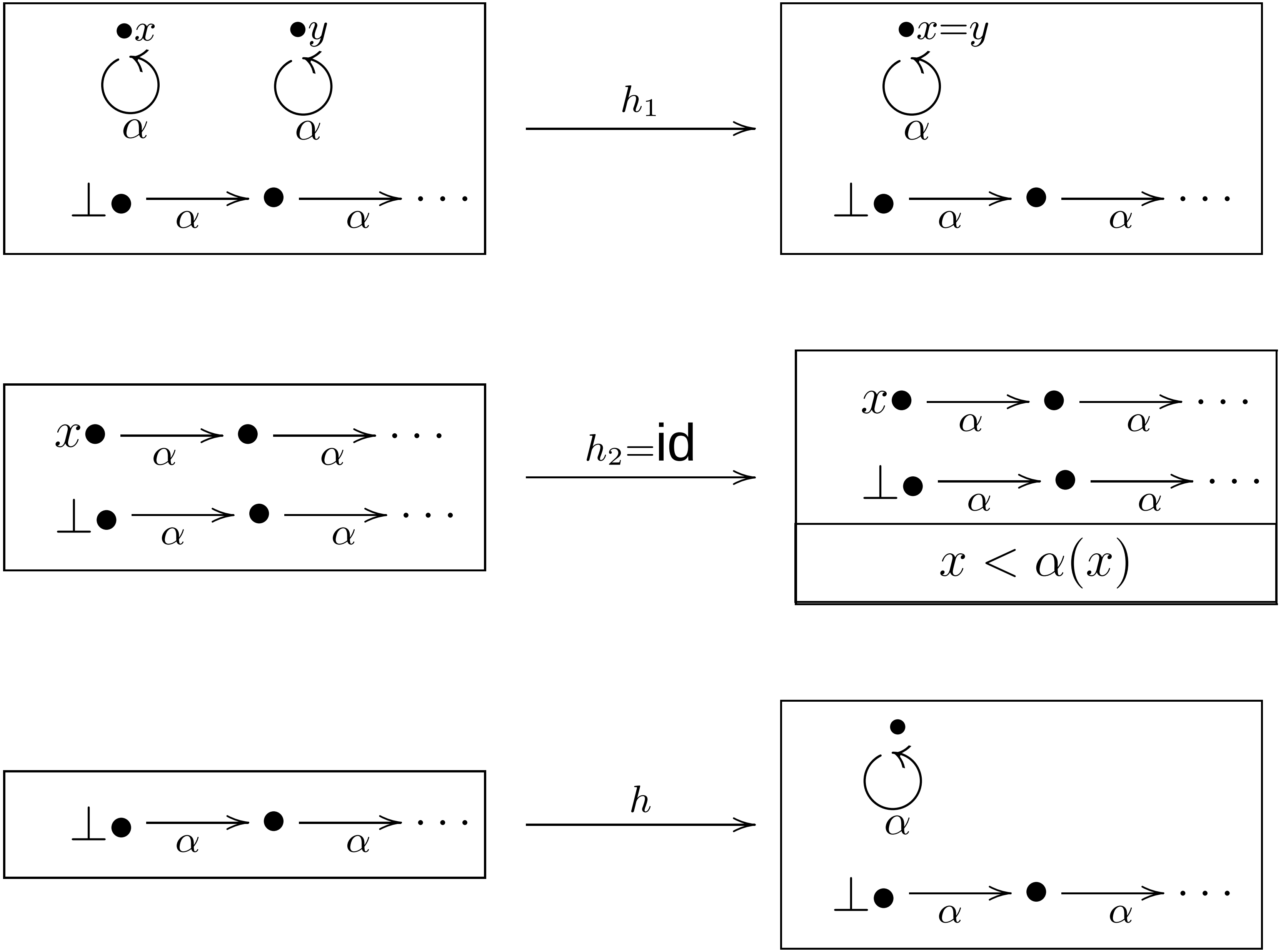

x
y
α
α
h1
x=y
α
⊥
α
α
⊥
α
α


x
y
α
α
h1
x=y
α
⊥
α
α
α
α
. . .
. . .

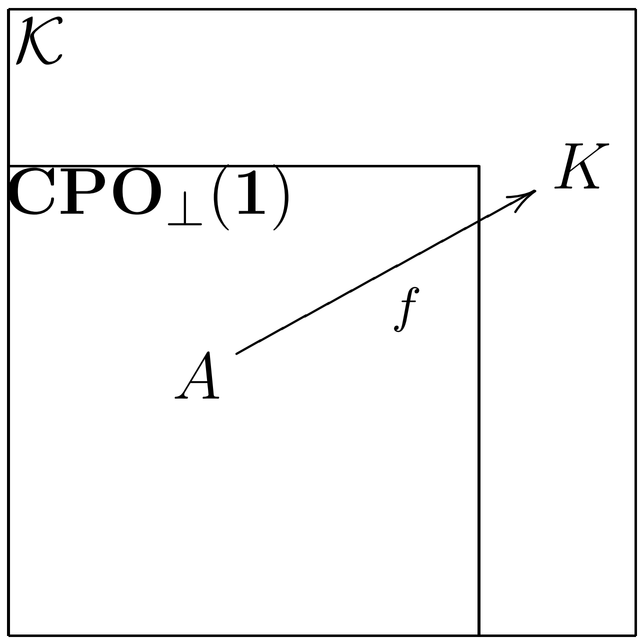

x
α
α
. . .
⊥
α
α
. . .
h2=id
x < α(x)



•x
•y
α
⊥
h₁
•x=y
x
h₂=id
x < α(x)
h



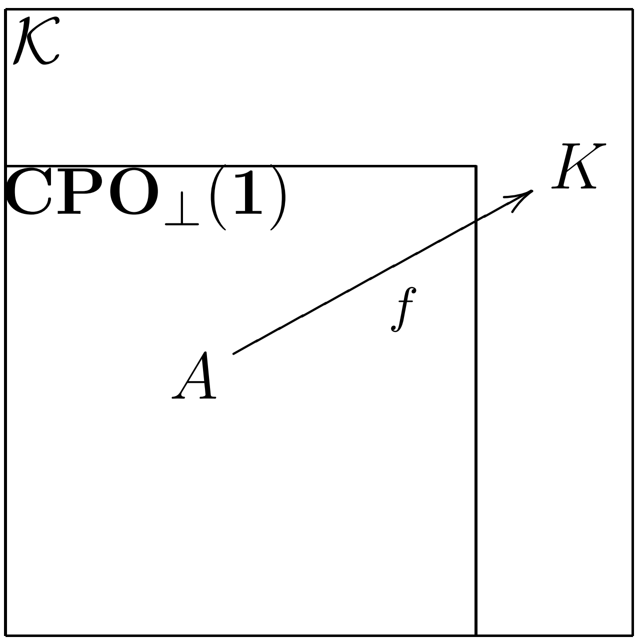
$$\{h_1,\, h_2\} \models h \text{ but } \{h_1,\, h_2\} \not\vdash h$$



$f : A \longrightarrow Ord$ is a *coloring* of A, that is:

f is continuous, $f(\perp) = 0$ and $f(\alpha(x)) = f(x) + 1$

$\mathcal{K}(A, K) = \{$colorings of $A\}$



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In $\mathbf{CPO}_\perp(\mathbf{1})$, $\{h_1, h_2\} \models h$

But in $\mathcal{K}$, K is orthogonal to $\{h_1, h_2\}$ but is NOT orthogonal to

Then: In $\mathbf{CPO}_\perp(\mathbf{1})$, $\{h_1, h_2\} \models h$, but $\{h_1, h_2\} \not\models h$.

In the Orthogonality Deduction System, for <u>sets</u> $\mathcal{H}$,

$$\mathcal{H} \models h \text{ iff } \mathcal{H} \vdash h$$

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Question: What about the completeness when we admit a proper class of morphisms $\mathcal{H}$ as premises?

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Spetial classes:

Classes of epimorphisms: Yes

Classes where just a set of morphisms are not epimorphisms: **??**

The completeness for classes of the Orthogonality Logic (in locally presentable categories) is equivalent to the Vopěnka's Principle.

existence of
huge cardinals

$\Downarrow$

Vopěnka's Principle $:= Ord$ has no full embedding into a loc. pres. cat.

$\Downarrow$

existence of
measurable cardinals