The functor category $\mathcal{F}_{quad}$ associated to quadratic spaces over $\mathbb{F}_2$

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Motivation : The category $\mathcal{F}$

Definition

$$\mathcal{F} = \mathrm{Funct}(\mathcal{E}^f, \mathcal{E})$$

$\mathcal{E}$: category of $\mathbb{F}_2$-vector spaces
$\mathcal{E}^f$: category of finite dimensional $\mathbb{F}_2$-vector spaces

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Example : Evaluation functors

$$\mathcal{F} \xrightarrow{E_n} \mathbb{F}_2[GL_n] - \mathrm{mod}$$

$$F \longmapsto F({\mathbb{F}_2}^n)$$

$\mathcal{F}$ and the stable cohomology of general linear groups

Let P and Q be two objects of $\mathcal{F} = \mathrm{Fonct}(\mathcal{E}^f, \mathcal{E})$

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{F}}(P,Q) \xrightarrow{E_n^*} \quad & \mathrm{Ext}^*_{\mathbb{F}_2[GL_n]-\mathrm{mod}}(P(\mathbb{F}_2{}^n), Q(\mathbb{F}_2{}^n)) \\ & = H^*(GL_n, \mathrm{Hom}(P(\mathbb{F}_2{}^n), Q(\mathbb{F}_2{}^n)))\end{aligned}$$

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Theorem (Dwyer)

If P and Q are finite (i.e. admit finite composition series),

$$\ldots \to H^*(GL_n, \mathrm{Hom}(P(\mathbb{F}_2{}^n), Q(\mathbb{F}_2{}^n))) \to H^*(GL_{n+1}, \mathrm{Hom}(P(\mathbb{F}_2{}^{n+1}), Q(\mathbb{F}_2{}^{n+1}))) \to \ldots$$

stabilizes. We denote by $H^*(GL, \mathrm{Hom}(P,Q))$ the stable value.

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Theorem (Suslin)

$$\mathrm{Ext}^*_{\mathcal{F}}(P,Q) \xrightarrow{\simeq} H^*(GL, \mathrm{Hom}(P,Q))$$

for P and Q finite

Aim

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$$O(H) \subset GL_{\dim(H)}$$

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Aim : Construct a "good" category $\mathcal{F}_{quad}$ related to orthogonal groups over $\mathbb{F}_2$

$$\mathcal{F}_{quad} \xrightarrow{E_H} \mathbb{F}_2[O(H)] - \mathrm{mod}$$

$$F \longmapsto F(H)$$

Preliminaries

V : finite $\mathbb{F}_2$-vector space

Definition

A quadratic form over V is a function $q : V \to \mathbb{F}_2$ such that

$$B(x, y) = q(x + y) + q(x) + q(y)$$

defines a bilinear form

Remark

The bilinear form B does not determine the quadratic form q

Definition

A quadratic space (V, q_V) is non-degenerate if the associated bilinear form is non singular

Properties of quadratic forms over $\mathbb{F}_2$

Lemma

The bilinear form associated to a quadratic form is alternating

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Classification of non-singular alternating bilinear forms

A space V equipped with a non-singular alternating bilinear form admits a symplectic base
i.e.$\{a_1, b_1, \ldots, a_n, b_n\}$ with $B(a_i, b_j) = \delta_{i,j}$ and $B(a_i, a_j) = B(b_i, b_j) = 0$

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Consequence :

A non-degenerate quadratic space (V, q_V) has even dimension

Classification of non-degenerate quadratic forms over $\mathbb{F}_2$

In dimension 2

There are two non-isometric quadratic spaces

$$
\begin{array}{rccc}
q_0 : & H_0 & \longrightarrow & \mathbb{F}_2 \\
 & a_0 & \longmapsto & 0 \\
 & b_0 & \longmapsto & 0 \\
 & a_0 + b_0 & \longmapsto & 1
\end{array}
\qquad
\begin{array}{rccc}
q_1 : & H_1 & \longrightarrow & \mathbb{F}_2 \\
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Proposition

$$H_0 \perp H_0 \simeq H_1 \perp H_1$$

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In dimension $2m$

There are two non-isometric quadratic spaces

$$H_0^{\perp m} \quad \text{and} \quad H_0^{\perp (m-1)} \perp H_1$$

The category $\mathcal{E}_q$

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- $\mathrm{Ob}(\mathcal{E}_q)$: non-degenerate quadratic spaces (V, q_V)
- morphisms are linear applications which preserve the quadratic form

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Replace $\mathcal{F} = \mathrm{Func}(\mathcal{E}^f, \mathcal{E})$ by $\mathrm{Func}(\mathcal{E}_q, \mathcal{E})$

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Natural Idea

Replace $\mathcal{F} = \mathrm{Func}(\mathcal{E}^f, \mathcal{E})$ by $\mathrm{Func}(\mathcal{E}_q, \mathcal{E})$

Proposition

Any morphism of $\mathcal{E}_q$ is a monomorphism

- $\mathcal{E}_q$ does not have enough morphisms : the category $\mathrm{Func}(\mathcal{E}_q, \mathcal{E})$ does not have good properties
- we seek to add orthogonal projections formally to $\mathcal{E}_q$

The category $\mathrm{coSp}(\mathcal{D})$ of Bénabou

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$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & D_1 \end{array}$$

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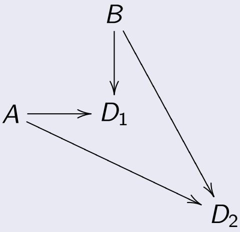

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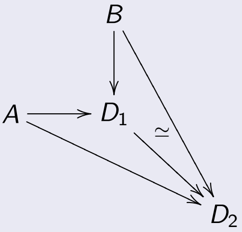

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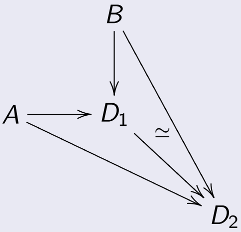

we denote by $[A \to D \leftarrow B]$ an element of $\mathrm{Hom}_{\mathrm{coSp}(\mathcal{D})}(A, B)$

Composition in the category $\mathrm{coSp}(\mathcal{D})$

$$\mathrm{Hom}_{\mathrm{coSp}(\mathcal{D})}(A, B) \times \mathrm{Hom}_{\mathrm{coSp}(\mathcal{D})}(B, C) \to \mathrm{Hom}_{\mathrm{coSp}(\mathcal{D})}(A, C)$$
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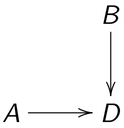

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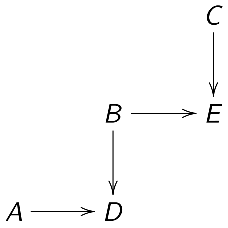

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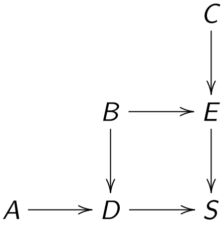

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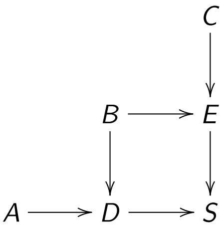

Dual construction : the category $\mathrm{Sp}(\mathcal{D})$

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Let $\mathcal{D}$ be a category equipped with pullbacks
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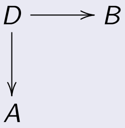

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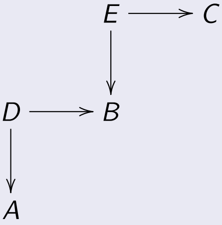

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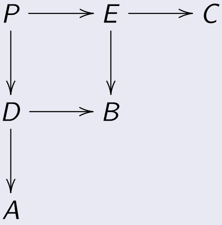

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For $f : V \to W$, let V' be the orthogonal complement of $f(V)$ in W
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We will write

$$f : V \to V \perp V'$$

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In the definition of $\mathrm{coSp}(\mathcal{D})$: universality of the push-out plays no role

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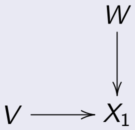

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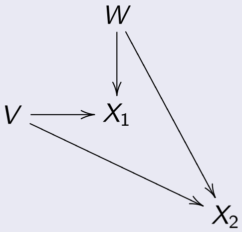

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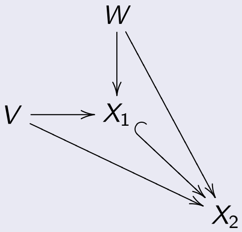

$\sim$: equivalence relation generated by this relation

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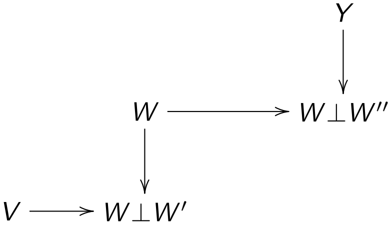

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$$\begin{array}{ccccc}
 & & & & Y \\
 & & & & \downarrow \\
 & & W & \longrightarrow & W \perp W'' \\
 & & \downarrow & & \downarrow \\
V & \longrightarrow & W \perp W' & \longrightarrow & W \perp W' \perp W''
\end{array}$$

Retractions in $\mathcal{T}_q$

Proposition

For $f : V \longrightarrow W$ a morphism of $\mathcal{E}_q$, we have :

$$[W \xrightarrow{\mathrm{Id}} W \xleftarrow{f} V] \circ [V \xrightarrow{f} W \xleftarrow{\mathrm{Id}} W] = \mathrm{Id}_V$$

that is $[W \xrightarrow{\mathrm{Id}} W \xleftarrow{f} V]$ is a retraction of $[V \xrightarrow{f} W \xleftarrow{\mathrm{Id}} W]$

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Theorem

The category $\mathcal{F}_{quad}$ is abelian, equipped with a tensor product and has enough projective and injective objects.

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Question

Classification of the simple objects of $\mathcal{F}_{quad}$

Reminder : A functor S is simple if it is not the zero functor and if its only subfunctors are 0 and S

The forgetful functor

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- On morphisms :

$$\epsilon([V \xrightarrow{f} W \perp W' \xleftarrow{g} W]) \quad = \quad p_g \circ f$$

where p_g is the orthogonal projection associated to g

Relating $\mathcal{F} = \mathrm{Funct}(\mathcal{E}^f, \mathcal{E})$ and $\mathcal{F}_{quad} = \mathrm{Funct}(\mathcal{T}_q, \mathcal{E})$

$$\mathcal{T}_q \xrightarrow{\epsilon} \mathcal{E}^f$$

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for F an object of $\mathcal{F} = \mathrm{Funct}(\mathcal{E}^f, \mathcal{E})$

Theorem

The functor $\iota : \mathcal{F} \to \mathcal{F}_{quad}$ defined by $\iota(F) = F \circ \epsilon$

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Consequence

$\mathrm{Sp}(\mathcal{E}_q^{\deg})$ is defined

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Definition

Iso_V is the functor of $\mathcal{F}_{iso}$ corresponding to $\mathbb{F}_2[O(V)]$ by this equivalence

Do we have all the simple objects of $\mathcal{F}_{quad}$?

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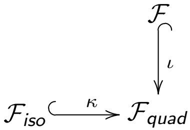

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- standard way to obtain a classification of the simple objects of a category : decompose the projective generators

IV Study of standard projective objects

Proposition (Yoneda lemma)

- *For V an object of $\mathcal{T}_q$, the functor defined by*

$$P_V(W) = \mathbb{F}_2[\mathrm{Hom}_{\mathcal{T}_q}(V, W)]$$

  is a projective object of $\mathcal{F}_{quad}$

- $\{P_V | V \in \mathcal{S}\}$ *: set of projective generators of $\mathcal{F}_{quad}$*
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Projective generators of $\mathcal{F}$

For E an object of $\mathcal{E}^f$

$$P_E^{\mathcal{F}}(X) = \mathbb{F}_2[\mathrm{Hom}_{\mathcal{E}^f}(E, X)]$$

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Notation

$\mathrm{Hom}_{\mathcal{T}_q}^{(i)}(V, W)$ the set of morphisms of $\mathrm{Hom}_{\mathcal{T}_q}(V, W)$ of rank $\leq i$

Rank filtration of the projective objects

Proposition

The functors $P_V^{(i)}$ *for* $i = 0, \ldots, \dim(V)$ *:*

$$P_V^{(i)}(W) = \mathbb{F}_2[\mathrm{Hom}_{\mathcal{I}_q}^{(i)}(V, W)]$$

define an increasing filtration of the functor P_V

$$0 \subset P_V^{(0)} \subset P_V^{(1)} \subset \ldots \subset P_V^{(\dim(V)-1)} \subset P_V^{(\dim(V))} = P_V$$

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1. $P_V^{(0)} \simeq \iota(P_{\epsilon(V)}^{\mathcal{F}})$ where $\iota : \mathcal{F} \to \mathcal{F}_{quad}$
2. *The functor* $P_V^{(0)}$ *is a direct summand of* P_V

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$$P_V / P_V^{(\dim(V)-1)} \simeq \kappa(\mathrm{Iso}_V)$$

where $\kappa : \mathcal{F}_{iso} \to \mathcal{F}_{quad}$

Decomposition of the functors P_{H_0} and P_{H_1}

$$0 \subset P^{(0)}_{H_\epsilon} \subset P^{(1)}_{H_\epsilon} \subset P_{H_\epsilon} \qquad \text{for } \epsilon \in \{0, 1\}$$

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Corollary

Classification of simple objects S of $\mathcal{F}_{quad}$ such that $S(H_0) \neq \{0\}$ or $S(H_1) \neq \{0\}$

The functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$

$\epsilon \in \{0,1\}$
(x,ϵ) : the degenerate quadratic space generated by x such that $q(x) = \epsilon$

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$\mathrm{Mix}_{\epsilon,1}$ *is isomorphic to a sub-functor of* $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{Iso}_{(x,\epsilon)})$

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Conjecture

Simple objects of $\mathcal{F}_{quad}$ are sub-quotients of tensor products between a simple functor of $\mathcal{F}$ and a simple functor of $\mathcal{F}_{iso}$