Galois theories of internal groupoids via congruence relations for Maltsev varieties

João J. Xarez

jxarez@mat.ua.pt

University of Aveiro
1 Coequalizer of the kernel pair

\( \mathbb{C} \) finitely-complete; \((F, \varphi)\) pointed endofunctor on \( \mathbb{C} \), s.t. the kernel pair of \( \varphi_A : A \to F(A) \) has a coequalizer for every object \( A \) in \( \mathbb{C} \).
2 Idempotency of \((I, \eta)\)

\(\text{Fix}(I, \eta), \text{Mono}(F, \varphi)\) full subcategories of \(\mathbb{C}\).

Lemma 2.1

\((I, \eta)\) well-pointed endofunctor (i.e., \(I\eta = \eta I\));
\(\text{Fix}(I, \eta) = \text{Mono}(F, \varphi)\).

Proposition 2.2

\(\mu, F\eta \text{ monics } \Rightarrow (I, \eta) \text{ idempotent}\)

Remark 2.3

\((I, \eta) \text{ idempotent } \iff I\eta = \eta I \text{ and } \eta I \text{ iso } \iff \text{Fix}(I, \eta) \text{ reflective in } \mathbb{C}\)
3 Stabilization and m.-l. factorization

Proposition 3.1

All $\eta_A$ pullback stable regular epis and $\mu$ monic and $F\eta$ iso and $F$ preserves

$$
\begin{array}{ccc}
C \times_{I(A)} A & \longrightarrow & A \\
\downarrow & & \downarrow \eta_A \\
C & \longrightarrow & I(A)
\end{array}
$$

$\Rightarrow (I, \eta)$ idempotent with stable units;

and $\forall B \in C \exists p : E \rightarrow B$ e.d.m. $E \in Mono(F, \varphi)$

$\Rightarrow (\mathcal{E}', \mathcal{M}^*)$ factorization system (monotone-light).
4 First example: internal categories

\((F, \varphi)\) idempotent associated to the localization

\[
\text{Cat}(S) \rightarrow \text{LEqRel}(S) \simeq S
\]

\[
C \mapsto \nabla_{C_0}
\]

\[
\begin{array}{ccc}
C &=& C_1 \times_{C_0} C_1 \\
\varphi_C &=& C_1 \xrightarrow{\gamma} C_1 \xleftarrow{i} C_0 \\
\nabla_{C_0} &=& C_0 \times C_0 \times C_0 \rightarrow C_0 \times C_0 \xrightarrow{1_{C_0}} C_0
\end{array}
\]
Lemma 4.1 $S$ regular $\Rightarrow$ for every $C \in \text{Cat}(S)$ the kernel pair of $\varphi_C = (d_C, 1_{C_0})$ has a coequalizer in $\text{Cat}(S)$.
Conclusion 4.2 $\mathcal{S}$ regular:

$\text{Cat}(\mathcal{S}) \rightarrow \text{Preord}(\mathcal{S})$ reflection with stable units;

$\text{Grpd}(\mathcal{S}) \rightarrow \text{EqRel}(\mathcal{S})$ reflection with stable units and monotone-light factorization,

$(\sigma, d_1) : Eq(d_0) \rightarrow G$, with $\sigma = \gamma(1_{G_1} \times s)$,

\[
\begin{array}{ccccccc}
G_1 \times_{G_0} G_1 & \xrightarrow{p_1 \times p_2} & G_1 \times_{G_0} G_1 & \xrightarrow{p_2} & G_1 \\
\downarrow \sigma \times \sigma & & \downarrow \sigma & & \\
G_1 \times_{G_0} G_1 & \xrightarrow{\gamma} & G_1 & \xrightarrow{i} & G_0
\end{array}
\]

$\sigma < 1_{G_1}, id_0 > = 1_{G_1}$ and $d_1 i = 1_{G_0}$. 

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e.g. $\mathcal{S} = \text{Set}$: $\text{Cat} \to \text{Preord}$,
$(\mathcal{E}', \mathcal{M}^*) = (\text{Full and Bijective on Objects, Faithful})$.

$\mathcal{S}$ Maltsev category: $\text{EqRel}(\mathcal{S}) = \text{RRel}(\mathcal{S}) (\Rightarrow \text{Cat}(\mathcal{S}) = \text{Grpd}(\mathcal{S}))$.

$\mathcal{S}$ regular Maltsev category: $\text{Grpd}(\mathcal{S}) \to \text{EqRel}(\mathcal{S}) = \text{RRel}(\mathcal{S})$
reflection with stable units and monotone-light-factorization.

A variety of universal algebras is Maltsev iff its theory has a Maltsev operator $p : X \times X \times X \to X$, $p(x, y, y) = x = p(y, y, x)$.

e.g. $\text{Grp}$:

$$p(x, y, z) = xy^{-1}z;$$

$$\text{Cat}(\text{Grp}) = \text{Grpd}(\text{Grp}) \simeq \text{CrossMod}.$$
5 Geometric morphisms

Corollary 5.1

$\mathcal{C}$ admits a (regular epi, mono)-factorization and $(F, \varphi)$ idempotent

$\Rightarrow (I, \eta)$ idempotent.

Corollary 5.2

$\mathcal{C}$ regular and $(F, \varphi)$ idempotent $\Rightarrow (I, \eta)$ idempotent;

and $F$ left exact $\Rightarrow$ stable units;

and $\forall B \in \mathcal{C} \exists p : E \rightarrow B$ e.d.m. $E \in \text{Mono}(F, \varphi) \Rightarrow m.-l. \text{ factorization.}$
Proposition 5.3 Let $F : \mathcal{E} \to \mathcal{F}$ be a geometric morphism between regular categories, $F^* \dashv F_* : \mathcal{E} \to \mathcal{F}$, which is an embedding.

Then, the reflection $I : \mathcal{F} \to \text{Mono}(F^*)$, obtained from the localization $F^* : \mathcal{F} \to \mathcal{E}$ through the coequalizer of the kernel pair process, does have stable units. Moreover, there is a monotone-light factorization associated to the reflection $I : \mathcal{F} \to \text{Mono}(F^*)$ provided the following four conditions also hold:

1. the category $\mathcal{F}$ is cocomplete;

2. the full subcategory $\text{Mono}(F^*)$ is dense in $\mathcal{F}$, i.e., every object of $\mathcal{F}$ is a colimit of objects of $\text{Mono}(F^*)$.

3. in $\mathcal{F}$ the coproduct of monomorphisms is a monomorphism;

4. regular epis are effective descent morphisms in $\mathcal{F}$. 
6 Second example: simplicial sets

\( K : B \to A \) fully faithful, \( S \) regular and complete

\[ S^K : S^A \to S^B \]

\( \Delta^{op}_n \subset \Delta^{op}, n \geq 0, S = \text{Set} \)

\[ \text{Smp} \to \text{Smp}_n \]

\[ \text{Smp} \to \text{Mono}(F_n) \]

\[ (F_n, \varphi^n) \mapsto (I_n, \eta^n) \]

**Lemma 6.1** Every unit morphism of any representable functor

\[ \varphi^n_{\Delta(-,[p])} : \Delta(-,[p]) \to F_n(\Delta(-,[p])), \ p \geq 0, \]

is a monomorphism in \( \text{Smp} = \text{Set}^{\Delta^{op}} \).