Can you Differentiate a Polynomial?

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WHAT IS THIS TALK ABOUT?

PART I: Differential Categories
PART II: Structural polynomials

Part II concerns one of the motivating example for the development of Cartesian Differential Categories!

... of course, examples can be very confusing.

$\otimes$-differential categories ...

- $\otimes$-Differential categories $=$ Seely category $+$ differential operator
- Simple categorical axiomatization
- Abstract framework for (additively enriched) differentiation
- Lots of sophisticated models
- Inspired by Ehrhard's work: Köthe spaces, finiteness spaces and (with Regnier) on the differential λ-calculus
- The "linear algebra" approach to calculus.

(developed with Blute and Seely)

$\times$-differential categories ...

$\otimes$-differential categories are not enough!

The following are in a *Cartesian* rather than *linear* world:

- Classical multivariable differential calculus ...
- Differential lambda calculus
  (Ehrhard, Renier, ... – French School)
- Combinatoric species differentiation
  (Joyal, Bergeron,.. – Montreal School)
- Differentiation of data types
  (McBride, Gahni, Fiori, .. – UK School)
- CoKleisli category of a $\otimes$-differential category ...

COMING SHORTLY ON A BIG SCREEN NEAR YOU:
THEIR CATEGORICAL AXIOMATIZATION!!

Many more differential categories ...

Aside: Cartesian differential categories are not enough either ...

- Classical differential calculus considers partial maps ...
- Calculus on manifolds uses topological notions ....
- Manifolds and varieties in algebraic geometry ...
- Synthetic differential geometry ..

HOW ARE THESE AXIOMATIZED?

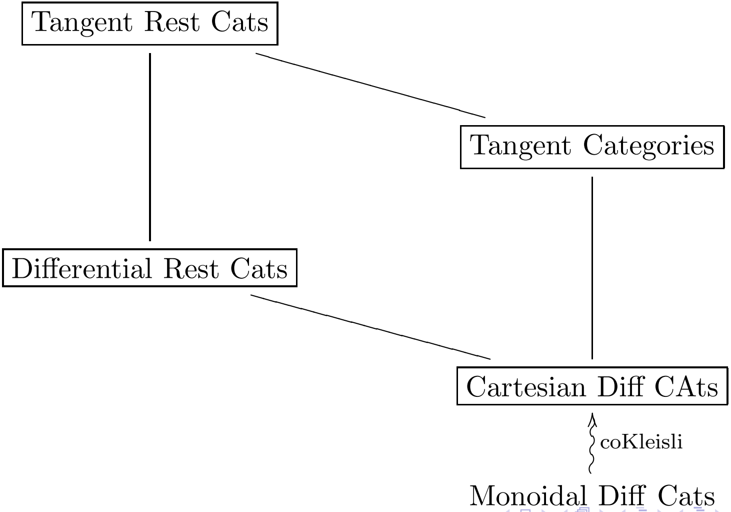
Many more differential categories ...

- Differential restriction categories (Cockett, Crutwell, Gallagher)
- Categories with tangent structure (Rosicky; Cockett, Crutwell)

Tangent structure is the most abstract of them all ...
.... and includes them all.
In particular, they provide link to synthetic differential calculus ....

One ring to rule them all ...

The story so far ...



Axioms for a cartesian differential ...

CARTESIAN DIFFERENTIAL CATEGORIES ARE ESSENTIALLY THE coKLEISLI CATEGORIES OF $\otimes$-DIFFERENTIAL CATEGORIES.

... just write down the equations ...

HOW HARD CAN THAT BE?

... generated two papers (and counting) so far!

It took a long time to get it right!

WHY?

A. We were idiots?
B. Academic baggage ...
C. Calculus for the masses ...
D. The structure of the area has been trampled on with:
    - Preconceptions: infinitesimals and dx ....
    - Manipulations without algebraic basis ...
    - Notational short-cuts which mask structure ...
E. The axioms are actually a little tricky!

DID WE GET IT RIGHT?

The good news:

We are confident we have *basic* axiomatization right!

FINALLY!

... and people are beginning to use it!

The bad news:

How do we know?

- capture coKleisli categories of diff cats
- captures key examples
- Faà di Bruno construction gives a comonadic description

... NOT EVERYONE IS CONVINCED

examples are very confusing!

.... effort needed to avoid reinventing the wheel!!

CARTESIAN DIFFERENTIAL CATEGORIES

To formulate cartesian differential categories need:

(a) Left additive categories

(b) Cartesian structure in the presence of left additive structure

(c) Differential structure

Example to have in mind: vector spaces with smooth functions ....

Left-additive categories

A category $\mathbb{X}$ is a **left-additive category** in case:

- Each hom-set is a commutative monoid $(0,+)$
- $f(g+h) = (fg) + (fh)$ and $f0 = 0$
  each f is **left additive** ..

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

A map h is said to be **additive** if it also preserves the additive structure on the right $(f+g)h = (fh) + (gh)$ and $0h = 0$.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$$

NOTE: additive maps are the exception ...

Lemma

In any left additive category:

(i) 0 *maps are additive;*

(ii) *additive maps are closed under addition;*

(iii) *additive maps are closed under composition;*

(iv) *identity maps are additive;*

(v) *if g is a retraction which is additive and the composite gh is additive then h is additive;*

(vi) *if f is an isomorphism which is additive then f^{-1} is additive.*

Additive maps form a subcategory ...

Example

(i) The category whose objects are commutative monoids CMon but whose maps need not preserve the additive structure.

(ii) Real vector spaces with smooth maps.

(iii) The coKleisli category for a comonad on an additive category when the functor is not additive.

Products in left additive categories

A **Cartesian left-additive category** is a left-additive category with products such that:

- the maps π_0, π_1, and Δ are additive;
- f and g additive implies $f \times g$ additive.

All our earlier example are Cartesian left-additive categories!

Lemma

The following are equivalent:

(i) *A Cartesian left-additive category;*

(ii) *A left-additive category for which $\mathbb{X}_+$ has biproducts and the the inclusion $\mathcal{I} : \mathbb{X}_+ \longrightarrow \mathbb{X}$ creates products;*

(iii) *A Cartesian category $\mathbb{X}$ in which each object is equipped with a chosen commutative monoid structure*

$$(+_A : A \times A \longrightarrow A, 0_A : 1 \longrightarrow A)$$

such that $+_{A\times B} = \langle (\pi_0 \times \pi_0)+_A, (\pi_1 \times \pi_1)+_B \rangle$ *and* $0_{A\times B} = \langle 0_A, 0_B \rangle$.

Lemma

In a Cartesian left-additive category:

(i) f is additive iff

$$(\pi_0 + \pi_1)f = \pi_0 f + \pi_1 f : A \times A \longrightarrow B \quad \textit{and} \quad 0f = 0 : 1 \longrightarrow B;$$

(ii) $g : A \times X \longrightarrow B$ is additive in its second argument iff

$$\begin{aligned} 1 \times (\pi_0 + \pi_1)g &= (1 \times \pi_0)g + (1 \times \pi_1)g : A \times X \times X \longrightarrow B \\ (1 \times 0)g &= 0 : A \times 1 \longrightarrow B. \end{aligned}$$

"Multi-additive maps" are maps additive in each argument.

A functor between Cartesian left-additive categories is **left-additive** in case $F(f+g) = F(f) + F(g)$ and $F(0) = 0$.

Lemma

A left-additive functor, $F : \mathbb{X} \longrightarrow \mathbb{Y}$, necessarily preserves products $F(A \times B) \equiv F(A) \times F(B)$, additive maps and multi-additive maps.

The category of all cartesian left-additive categories and left-additive functors is CLAdd.

An operator $D_\times$ on the maps of a Cartesian left-additive category

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D_\times[f]]{} Y}$$

is a **Cartesian differential operator** in case it satisfies:

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;

[CD.2] $\langle (h+k), v\rangle D_\times[f] = \langle h, v\rangle D_\times[f] + \langle k, v\rangle D_\times[f]$;

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$;

[CD.4] $D_\times[\langle f, g\rangle] = \langle D_\times[f], D_\times[g]\rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$.

[CD.6] $\langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$;

[CD.7] $\langle\langle 0, f\rangle, \langle g, h\rangle\rangle D_\times[D_\times[f]] = \langle\langle 0, g\rangle, \langle f, h\rangle\rangle D_\times[D_\times[f]]$

A Cartesian left-additive category with such a differential operator is a **Cartesian differential category**.

What was so hard about that?

ANSWER: the last two rules!!

- They are independent ...
- They involve higher differentials ...
- Not so obvious where they come from ...

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;
(operator preserves additive structure)

[CD.2] $\langle (h+k), v\rangle D_\times[f] = \langle h, v\rangle D_\times[f] + \langle k, v\rangle D_\times[f]$
(always additive in first argument);

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$
(coherence maps are linear);

[CD.4] $D_\times[\langle f, g\rangle] = \langle D_\times[f], D_\times[g]\rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$)
(operator preserves pairing);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$ (chain rule);

[CD.6] $\langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$
(differentials are linear in first argument);

[CD.7] $\langle\langle 0, f\rangle, \langle g, h\rangle\rangle D_\times[D_\times[f]] = \langle\langle 0, g\rangle, \langle f, h\rangle\rangle D_\times[D_\times[f]]$
(partial differentials commute);

Real vector spaces with smooth maps are the "standard" example of a Cartesian differential category.

$$\dfrac{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, .., x_n) \\ \vdots \\ f_m(x_1, .., x_n) \end{pmatrix}}{\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \right) \mapsto \begin{pmatrix} \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \\ \vdots \\ \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \end{pmatrix}} \; \mathrm{D}$$

Polynomials are an example:

The category $\mathrm{Poly}(\mathbb{N})$:

Objects: The natural numbers: $0, 1, 2, 3, \ldots$

Maps: $(p_1, \ldots, p_n) : m \longrightarrow n$ where $p_i \in \mathbb{N}[x_1, .., x_m]$

Composition: By substitution.

This is the Lawvere theory of commutative rigs ...

The differential is:

$$\frac{m \longrightarrow n; (x_1, .., x_m) \mapsto (p_1, .., p_n)}{(\sum_i y_i \cdot \partial_i p_1, \ldots, \sum_i y_i \cdot \partial_i p_n) : m + m \longrightarrow n}$$

Not the polynomials of this talk!

Given a $\times$-differential category $\mathbb{X}$ the simple slice, $\mathbb{X}[A]$, at any object A is a differential category.

... think of A as giving a context!

$\mathbb{X}[A]$ is defined as:

Objects: $X \in \mathbb{X}$ as before;

Maps: $f : X \longrightarrow Y$ in $\mathbb{X}[A]$ are maps $f : X \times A \longrightarrow Y$ in $\mathbb{X}$;

Composition: fg in $\mathbb{X}[A]$ is $\langle f, \pi_1 \rangle g$ in $\mathbb{X}$;

Identities: 1_X in $\mathbb{X}[A]$ is $\pi_0 : X \times A \longrightarrow X$ in $\mathbb{X}$;

Differential $D_\times[\_]$ in $\mathbb{X}[A]$ is $(\langle 1, 0 \rangle \times 1) D_\times[f]$ in $\mathbb{X}$.

The differential in the simple slice is the "partial" derivative in the original.

Partial derivatives are important!

Linear maps

A map in a $\times$-differential category is **linear** in case:

$$D_\times[f] = \pi_0 f$$

Lemma

(i) *Linear maps are additive;*
(ii) *Identities and projections are linear;*
(iii) *If f and g are linear then $\langle f, g\rangle$ is linear;*
(iv) *Linear maps compose.*

SO linear maps form an additive subcategory which includes products.

Linear maps

A map $f : A \times B \longrightarrow C$ is linear in its first argument if

$$(\langle 1, 0\rangle \times 1)D[f] = \langle \pi_0, \pi_1\pi_1\rangle f : A \times (A \times B) \longrightarrow C$$

Equivalently $f \in \mathbb{X}[B]$ is linear. This gives **multi-linear maps** ....

NOTE: every multi-linear map is multi-additive BUT the converse is not true in general.

Linear maps play a key role in the structure of Cartesian differential categories (they form a *linear system of maps*.

Linear maps

Recall:

$$[\mathbf{CD.6}] \quad \langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$$

This say differentials are linear in their first argument – already know they are additive by **[CD.2]**.

In vector spaces with smooth maps $f : K^n \longrightarrow K^m$ is linear if and only if it is a linear transformation in the usual sense.
In polynomials $(f_1, ..f_n)$ is linear if and only if each f_i is of the form

$$a_1 \cdot x_1 + ... + a_m \cdot x_m$$

no constants no higher-order terms.

Term logic

Any Cartesian category has a *term logic* which is simply a typed version of usual equational logic. The differential can be added:

$$\frac{x : X, \Gamma \vdash t : Y \quad \Gamma \vdash p : X \quad \Gamma \vdash u : X}{\Gamma \vdash \frac{\mathrm{d}x}{\mathrm{d}t}(p) \cdot u}$$

This is read as: "the differential of t with respect to x at position p acting on the vector u".
NOTE: the differential term *binds* the variable x ....
This is interpreted as the *partial* differential:

$$\Gamma \xrightarrow[\langle [\![u]\!] , \langle [\![p]\!] , \pi_1 \rangle (\langle 0, 1 \rangle \times 1) D_\times [\![t]\!]]{} Y$$

Term logic

The basic structural judgements for Cartesian categories:

$$\frac{}{\Gamma, x : T \vdash x : T}\ \text{Proj} \qquad \frac{\Gamma \vdash t' : T'}{\Gamma, x : T \vdash t' : T'}\ \text{Weak}$$

$$\frac{\Gamma \vdash t' : T'}{\Gamma, () : 1 \vdash t' : T'}\ \text{Unit} \qquad \frac{\Gamma, x : T_1, y : T_2 \vdash t' : T'}{\Gamma, (x, y) : T_1 \times T_2 \vdash t' : T'}\ \text{Pair}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2}\ \text{Tuple} \qquad \frac{}{\Gamma \vdash () : 1}\ \text{UnitTuple}$$

Term logic

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1 + t_2 : T}\ \text{Add} \qquad \frac{}{\Gamma \vdash 0 : T}\ \text{Zero}$$

$$\frac{\{\Gamma \vdash t_i : T_i\}_{i=1,..,n} \quad f \in \Omega(T_1, ..., T_n; T)}{\Gamma \vdash f(t_1, ..., t_n) : T}\ \text{Fun}$$

$$\frac{\Gamma, x : S \vdash t : T \quad \Gamma \vdash p : S \quad \Gamma \vdash u : S}{\Gamma \vdash \frac{\mathrm{d}t}{\mathrm{d}x}(p) \cdot u : T}\ \text{Diff}$$

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma, x : T \vdash t_2 : T'}{\Gamma \vdash t_2[t_1/x] : T'}\ \text{Cut}$$

Note the differential term is a “binding” or a ”quantification”. Not viewed as an infinitesimals ....

Term logic

[Dt.1] $\frac{\mathrm{d}(t_1+t_2)}{\mathrm{d}x}(p)\cdot u = \frac{\mathrm{d}t_1}{\mathrm{d}x}(p)\cdot u + \frac{\partial p}{\partial t_2}(x)\cdot u$ and $\frac{\mathrm{d}0}{\mathrm{d}x}(p)\cdot u = 0$;

[Dt.2] $\frac{\mathrm{d}t}{\mathrm{d}x}(p)\cdot(u_1+u_2) = \frac{\mathrm{d}t}{\mathrm{d}x}(p)\cdot u_1 + \frac{\partial p}{\partial t}(x)\cdot u_2$ and $\frac{\mathrm{d}t}{\mathrm{d}x}(p)\cdot 0 = 0$;

[Dt.3] $\frac{\mathrm{d}x}{\mathrm{d}x}(p)\cdot u = u$, $\frac{\mathrm{d}t}{\mathrm{d}(x_1,x_2)}(p_1,p_2)\cdot(u_1,0) = \frac{\mathrm{d}t[p_2/x_2]}{\mathrm{d}x_1}(p_1)\cdot u_1$ and $\frac{\mathrm{d}t}{\mathrm{d}x_1,x_2}(p_1,p_2)\cdot(0,u_2) = \frac{\mathrm{d}t[p_1/x_1]}{\mathrm{d}x_2}(p_2)\cdot u_2$;

[Dt.4] $\frac{\mathrm{d}(t_1,t_2)}{\mathrm{d}x}(p)\cdot u = \left(\frac{\mathrm{d}t_1}{\mathrm{d}x}(p)\cdot u, \frac{\mathrm{d}t_2}{\mathrm{d}x}(p)\cdot u\right)$;

Term logic

[Dt.5] $\frac{\mathrm{d}t[t'/y]}{\mathrm{d}x}(p)\cdot u = \frac{\mathrm{d}t}{\mathrm{d}y}(t'[p/x])\cdot\left(\frac{\mathrm{d}t'}{\mathrm{d}x}(p)\cdot u\right)$
(The chain rule: no variable of p occur in t);

[Dt.6] $\frac{\mathrm{d}\frac{\mathrm{d}t}{\mathrm{d}y}(p')\cdot x}{\mathrm{d}x}(p)\cdot u = \frac{\mathrm{d}t}{\mathrm{d}y}(p')\cdot u.$

[Dt.7] $\frac{\mathrm{d}\frac{\mathrm{d}t}{\mathrm{d}x_1}(p_1)\cdot u_1}{\mathrm{d}x_2}(p_2)\cdot u_2 = \frac{\mathrm{d}\frac{\mathrm{d}t}{\mathrm{d}x_2}(p_2)\cdot u_2}{\mathrm{d}x_1}(p_1)\cdot u_1$
(Independence of partial derivatives: s_1, u_1, s_2, u_2 do not contain variables from x_1 or x_2)

The term logic is standard calculus!

Term logic

Can use the term logic to freely add a differential to a left additive category: so

$$U : \mathrm{CartDiff} \longrightarrow \mathrm{CLAdd}$$

has a left adjoint ...

Interestingly U also has a right adjoint! Given by Faà di Bruno ....

Faà di Bruno Francesco Faà di Bruno (1825-1888) was an Italian of noble birth, a soldier, a mathematician, and a priest. In 1988 he was beatified by Pope John Paul II for his charitable work teaching young women mathematics. As a mathematician he studied with Cauchy in Paris. He was a tall man with a solitary disposition who spoke seldom and, when teaching class, not always successfully. Perhaps his most significant mathematical contribution concerned the combinatorics of the higher-order chain rules. These results where the cornerstone of "combinatorial analysis": a subject which never really took off.

Our interest is in the higher-order chain rule ...

Faà di Bruno Higher-order derivatives are defined recursively:

$$
\begin{aligned}
\frac{\mathrm{d}^{(1)}t}{\mathrm{d}x}(p) \cdot u &= \frac{\mathrm{d}t}{\mathrm{d}x}(p) \cdot u \\
\frac{\mathrm{d}^{(n)}t}{\mathrm{d}x}(p) \cdot u_1 \cdot \ldots \cdot u_n &= \frac{\mathrm{d}\frac{\mathrm{d}^{(n-1)}t}{\mathrm{d}x}(x) \cdot u_1 \cdot \ldots \cdot u_{n-1}}{\mathrm{d}x}(p) \cdot u_n
\end{aligned}
$$

QUESTION:
What do the higher-order chain rule look like?

$$
\frac{\mathrm{d}^{(n)}g(f(x))}{\mathrm{d}x}(p) \cdot u_1 \cdot \ldots \cdot u_n = ????
$$

The answer involves some combinatorics ...

Faà di Bruno

A **symmetric tree** of depth $n \geq 0$ and in variables $x_1, \ldots, x_m$ is:

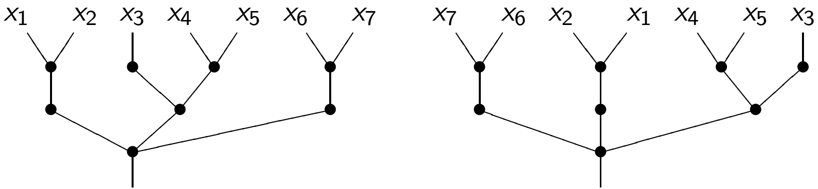
- The only symmetric tree of height 0 has width 1 and is a variable y;
- A symmetric tree of height $n \geq 1$ in the variables $x_1, \ldots, x_m$, that is of width m, is an expression $\bullet_r(t_1, \ldots, t_r)$ where each t_i is a symmetric tree of height $n-1$ in the variables X_i, where $\bigsqcup_{i=1}^{r} X_i = X$.

Note that the inductive step involves splitting the variables into r disjoint non-empty subsets. The combinatorics of this is described by Stirling numbers, of the second kind.

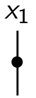
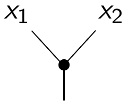
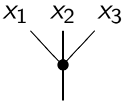
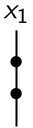
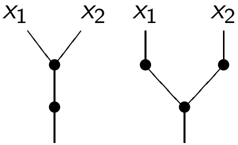
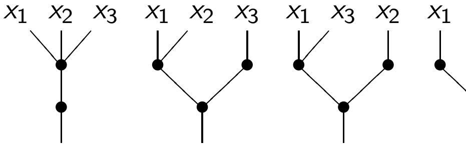
The operations at the nodes are viewed as being **symmetric**, or commutative:

$$\bullet_r(t_1, \ldots, t_r) = \bullet_r(t_{\sigma(1)}, \ldots, t_{\sigma(r)})$$

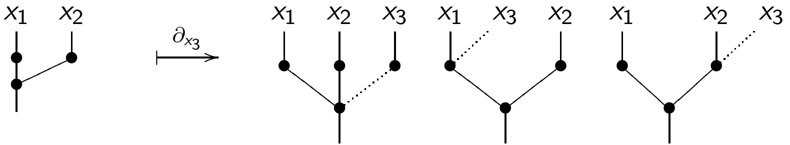
Faà di Bruno Here are two representations of the same symmetric tree:



Faà di Bruno A classification of the first few symmetric by height and width:

	width 1	width 2	width 3
hgt 1			
hgt 2			

Faà di Bruno The differential of a symmetric tree τ of height n and width r produces a bag of m trees of height n and width $r+1$, where m is the number of nodes of τ. The new trees of the differential are produced by picking a node and adding a "limb" to the new variable. The limb consists of a series of unary nodes applied to the new variable: the unary nodes retain the uniform height of the tree.



All symmetric trees of a given height and width can be obtained by differentiating the unique tree of width one of the same height, ι_h.

Faà di Bruno

The Faà di Bruno (bundle) category, Faà($\mathbb{X}$).

Objects: pairs of objects of the original category (A, X) (diagonal case (A, A));

Maps: $f : (A, X) \longrightarrow (B, Y)$ are infinite sequences of *symmetric forms*

$$f = (f_*, f_1, f_2, ...) : (A, X) \longrightarrow (B, X)$$

Where $f_* : X \longrightarrow Y$ is a map in $\mathbb{X}$ and, for $r > 1$,

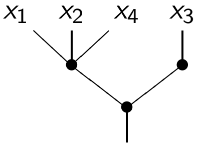
$$f_r : \underbrace{A \times ... \times A}_{r} \times X \longrightarrow B$$

is additive in each of the first r arguments and symmetric in these arguments.

Identities: $(1, \pi_0, ...) : (A, X) \longrightarrow (A, X)$

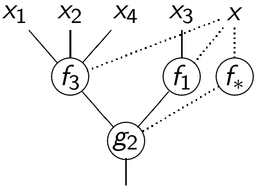
Composition: Faà di Bruno convolution ...

Faà di Bruno Faà di Bruno convolution ... when τ is the following tree



then

$$(f,g)\star\tau(x) = (((x_1,x_2,x_4,z)f_3,(x_3,z)f_1,f_*(x))g_2 : \underbrace{A\times \ldots \times A}_{4} \times X \longrightarrow C.$$



Notice that $(f,g)\star\tau(x)$ is additive in each argument except the

Faà di Bruno Faà di Bruno convolution:

$$(fg)_n = \sum_{\tau \in \mathcal{T}_2^n} (f, g) \star \tau$$

where $\mathcal{T}_2^n$ is all symmetric trees of height 2 and width n. This gives an associative composition with unit.

Observations:

- Faà : CLAdd $\longrightarrow$ CLAdd is a functor;
- ε : Faà$(\mathbb{X}) \longrightarrow \mathbb{X}; (f_*, f_1, f_2, \ldots) \mapsto f_*$ is a fibration and a natural transformation in CLAdd;
- A differential Cartesian category has a section to this fibration: $f \mapsto (f, f^{(1)}, f^{(2)}, \ldots)$

Faà di Bruno
In fact:

Theorem

Faà : CLAdd $\longrightarrow$ CLAdd *gives a comonad on* CLAdd *which (when restricted to diagonal objects) has coalgebras which are exactly cartesian differential categories.*

More proof of the pudding ...

End of part I

- Some basic examples of differential categories;
- The term logic
- The Faà di Bruno construction.

References

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