More work for Robin
Supplementary Handout
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DEFINITIONS

A **Boolean restriction category** is a split restriction category with

- finite coproducts.
- 0 is a zero object.
- The class of monics arising from splitting restriction idempotents is the coproduct injections.
- If $f, g : X \to Y$ with $f \perp g$ (which means $\overline{f}\,\overline{g} = 0$) then $f \vee g$ exists and $u(f \vee g)t = uft \vee ugt$.

A category is a Boolean restriction category if and only if it is isomorphic to the partial morphism category of an extensive category, with coproduct injections for the stable class of monics.

A category is **preadditive** if

- $X + Y$ exists.
- 0 is a zero object.
- Given a coproduct $P \xrightarrow{i} X \xleftarrow{i'} P'$, the "projections" $P \xleftarrow{\rho} X \xrightarrow{\rho'} P'$ defined by
$$\rho = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \rho' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
are jointly monic. $f, g : X \to Y$ are **summable** if there exists $t : X \to Y + Y$ with $\rho_1 t = f$, $\rho_2 t = g$ in which case their **sum** is $f + g = \begin{pmatrix} f \\ g \end{pmatrix} t$.

A **semiadditive category** is a preadditive category in each each two $f, g : X \to Y$ are summable. In that case, hom-sets are abelian monoids, and ρ, ρ' are the projections of a product. See [1], [30, Section I.18], [34, Section 12.2].

An **action** of a Boolean algebra B on an abelian monoid $(A, +, 0)$ is $B \times A^2 \to A$ satisfying

(BA.1) $1(f, g) = f$

(BA.2) $p'(f, g) = p(g, f)$

(BA.3) $pq(f, g) = p(q(f, g), g)$

(BA.4) $p(f + g, t + u) = p(f, t) + p(g, u)$

(BA.5) If $pq = 0$, $p(f, g(f, 0)) = p(f, 0) + q(f, 0)$

A **McCarthy algebra** is $(M, \vee, \wedge, (\cdot)', 0, 2)$ subject to

(M.1) $x'' = x$

(M.2) $(x \wedge y)' = x' \vee y'$

(M.3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

(M.4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(M.5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$

(M.6) $x \vee (x \wedge y) = x$

(M.7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$

(M.8) $0 \wedge x = x,\ 2 \wedge x = 2$

(M.9) $2' = 2,\ 0' \wedge 2 = 2,\ 0 \wedge 2 = 0$

The McCarthy algebra $3 = \{0, 1, 2\}$ with

x	x'
0	1
1	0
2	2

$x \wedge y$	0	1	2
0	0	0	0
1	0	1	2
2	2	2	2

$x \vee y$	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

generates the variety of McCarthy algebras, so truth-table analysis in 3 can be used to verify any potential equation of McCarthy algebras. 3 is the only subdirectly irreducible McCarthy algebra (as defined below).

Let B be a Boolean algebra. Let M_B be the set of all pairs (p, q) with $p, q \in B$, $p \wedge q = 0$. Define

$$\begin{aligned} 0 &= (0,1) \\ 2 &= (0,0) \\ (p,q)' &= (q,p) \\ (p,q) \wedge (r,s) &= (p \wedge q, q \vee (p \wedge s)) \\ (p,q) \vee (r,s) &= (p \vee (q \wedge r), q \wedge s) \end{aligned}$$

Then M_B is a McCarthy algebra.

The origin of the idea is simple. There is a natural bijection between 3^I and pairs of disjoint subsets of I via

$$I \xrightarrow{f} 3 \quad \mapsto \quad (f^{-1}0, f^{-1}1)$$

The formulas above are the transport of the pointwise operations in 3^I.

A **subdirect embedding** of algebra A in a family $\mathcal{B}$ of algebras is a subalgebra $A \to \prod B_i$ with all $B_i \in \mathcal{B}$ and all $A \to \prod B_i \xrightarrow{pr_j} B_j$ surjective.

A is **subdirectly irreducible** if $|A| > 1$ and A admits no non-trivial subdirect embedding, i.e. if $A \to \prod B_i$ is subdirect, some $A \to \prod B_i \xrightarrow{pr_j} B_j$ is an isomorphism.

In 1935, Garrett Birkhoff proved:

Proposition A is subdirectly irreducible if and only if the intersection of all non-diagonal congruences on A is again non-diagonal.

Proof idea If $\mathcal{R}$ is the set of all non-diagonal congruences, consider the canonical map $A \to \prod_{R\in\mathcal{R}} A/R$.

Algebra A is **primal** if A is finite with at least two elements and is such that for all $n > 0$, every function $A^n \to A$ is the interpretation of some term.

It is well known, indeed is a staple of electrical engineering, that 2 is primal in the variety of Boolean algebras.

Theorem (Krauss, 1942) Let P be a primal algebra.

- Each finite algebra in the variety $Var(P)$ generated by P is isomorphic to P^m for some m.
- P is the only primal algebra in $Var(P)$.
- Two varieties each generated by a primal algebra of the same cardinality are isomorphic.

EXERCISES

1. In a Boolean restriction category $\mathcal{B}$, show that $\mathcal{B}$ is preadditive, and that $f, g : X \to Y$ are summable if and only if $f \perp g$, i.e. $\overline{f}\,\overline{g} = 0$. If f, g are summable show that $f + g = f \vee g$. Hint. Show $\rho_1 \perp \rho_2$ and that $t = in_1 f \vee in_2 g$. You will need some basic facts from Cockett and Manes 2009.

2. A semigroup is **left zero** if $xy = x$ and **right zero** if $xy = y$. In a semigroup, two of Green's relations are $x\,\mathcal{L}\,y$ if there exists t, u with $tx = y$ and $uy = x$; $x\,\mathcal{R}\,y$ if there exists t, u with $xt = y$, $yu = x$. Prove that the following statements are equivalent (these define a **rectangular band**).

   (a) $xyx = x$
   (b) $x^2 = x,\ xyz = xz$
   (c) $x\,\mathcal{L}\,y \Leftrightarrow x = xy\,;\, x\,\mathcal{R}\,y \Leftrightarrow y = xy$
   (d) If $xy = yx$ then $x = y$.
   (e) $S \cong L \times R$ with L left zero and R right zero.

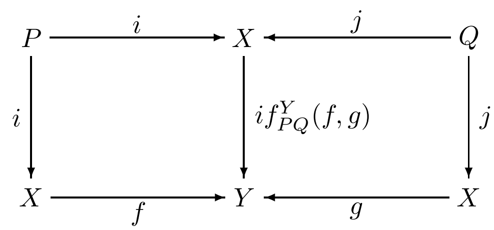
   Hints. For (a)⇒(b), $xyz = xy(zxz) = ...$ For (b)⇒(c), if $x = ty$ then $x = xy$ and $y = yx$. For (d)⇒(e), $\mathcal{L}$ and $\mathcal{R}$ are semigroup congruences (true here, but not generally in a semigroup). The canonical map $X \to X/\mathcal{L} \times X/\mathcal{R}$ is an isomorphism. To prove surjective, given x, y one has $x\,\mathcal{R}\,xy\,\mathcal{L}y$.

3. Let B be a Boolean algebra acting on an abelian monoid. Prove the following.

   (a) Each $p \in B$ is **total**, that is, $p(f, f) = f$.
   (b) Defining $pf = p(f, 0)$, show $p(f, g) = pf + p'g$.
   (c) $p(\cdot, \cdot)$ is a rectangular band.
   (d) Say that binary operations a, b **commute** if $a(b(f, g), b(t, u)) = b(a(f, t), a(g, u))$. Show that $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$ commute for every $p, q \in B$.

4. Let A be an abelian semigroup.
   (a) Show that $x \leq y$ if $x^2 = xy$ is a partial order if and only if $x^2 = xy = y^2 \Rightarrow x = y$.
   (b) Suppose further that $\forall x \in A\ \exists n > 1\ x^n = x$. Show that A is an inverse semigroup whose restriction order (under the restriction $\overline{x} = x^{-1}x$) coincides with $x^2 = xy$ as above.

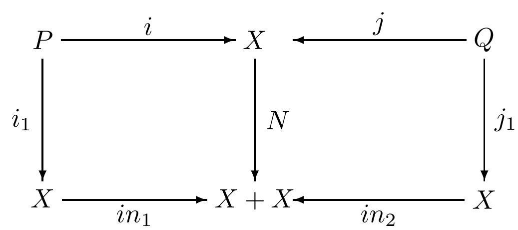
5. For $P \xrightarrow{i} X \xleftarrow{j} Q$ a coproduct in a category $\mathcal{X}$, define maps $if^Y_{PQ}(f,g)$ by



   (a) Show that each if^Y_{PQ} is a rectangular band and that if^Y_{PQ} is natural in Y

$$\mathcal{X}(X,\cdot) \times \mathcal{X}(X,\cdot) \to \mathcal{X}(X,\cdot)$$

   (b) Assume that $\mathcal{X}$ has binary copowers and assume that given a split monic $N : X \to X+X$ the pullbacks



   exist with the top row a coproduct. For fixed X, let

$$I : \mathcal{X}(X,\cdot) \times \mathcal{X}(X,\cdot) \to \mathcal{X}(X,\cdot)$$

   be a natural transformation which is pointwise a rectangular band. Show that a coproduct $P \xrightarrow{i} X \xleftarrow{j} Q$ exists with $I = if_{PQ}$. Hint. I corresponds to a map $N : X \to X + X$ by Yoneda. By rectangular band, the codiagonal $X + X \to X$ is a common splitting of in_1 and N so that $i = i_1$, $j = j_1$.

A **Boolean ring** is a ring (not necessarily with unit) in which $x^2 = x$. Thus a Boolean algebra is a Boolean ring with unit –finite subsets of an infinite set is a Boolean ring which is not a Boolean algebra.

6. (a) Show that a Boolean ring is commutative with $-x = x$. Hint. Consider $(x+y)^2 = x+y$.
   (b) For R a Boolean ring, $y \in R$, show that $R_y = \{xy + x : y \in R\}$ is a subring.
   (c) Show that 2 is the only subdirectly irreducible Boolean ring. Hint. Suppose $0 < x < y$ in R. Then $\psi : R \to [0,y] \times R_y$, $\psi x = (xy, xy + x)$ is a subdirect embedding.

7. A **3-ring** is a commutative ring satisfying $3x = 0$, $x^3 = x$. In a 3-ring, prove the identity

$$x \;=\; 1 + (x - x^2) + ((x+2)^2 - (2+x))$$

Conclude that every element is the sum of three idempotents.

8. Show directly from the axioms on a McCarthy algebra that the following duality principle holds: given any true equation of McCarthy algebras, the equation resulting from interchanging $\wedge$ and $\vee$ also remains true.

9. In a McCarthy algebra M, define $x + y = (x \wedge y') \vee (x' \wedge y)$. (In a Boolean algebra, this would be symmetric difference; here, the order of x, y' and x', y is crucial). Show that $(M, +, 0)$ is an abelian monoid and that $x \wedge (y + z) = (x \wedge y) + (x \wedge z)$.

10. Let M be a McCarthy algebra. For $a \in M$ define $x \, \theta_a \, y$ to mean $a \wedge x = a \wedge y$. Define $[0, a] = \{a \wedge x : x \in M\}$.
    (a) Show that θ_a is a congruence. Hint. Use $a \wedge (a' \vee x) = x$.
    (b) Show that θ_a is nontrivial if and only if $0 \neq a \neq 1$.
    (c) Show that the composition $[0, a] \subset M \to M/\theta_a$ is a bijection, rendering $[0, a]$ a McCarthy algebra with the same $0, \wedge, \vee$ and with complement $a \wedge x'$ and $2 = a \wedge 2$.

11. Prove that a primal algebra P has no nontrivial subalgebras or quotient algebras. Hints. If S were a nontrivial subalgebra, consider a function $P \to P$ not mapping S into itself. If R is a congruence with $x \, R \, y$ but $x \neq y$ and if $u, v \in P$ are arbitrary, consider $f : P \to P$ with $fx = u$, $fy = v$.

12. Prove that every finite McCarthy algebra is a subalgebra of M_B for some finite Boolean algebra B. Hint. By the proof of Birkhoff's theorem above, each finite algebra is subdirectly embeddable in a *finite* product of subdirect irreducibles. If $B = 2^n$ then $M_B = 3^n$.

CHALLENGES FOR RESTRICTION CATEGORIES

Restriction categories abstract the category **Pfn** of sets and partial functions, the natural universe in which to discuss deterministic computation. When one is willing to allow multithreaded computation in which one input may give eventual rise to different outputs, it is usually assumed without much thought that **Pfn** generalizes to **Rel**, the category of sets and relations. Just as **Pfn** is the springboard example for restriction categories, **Rel** motivates and abstracts to allegories [13]. Like restriction categories, allegories are a varietal extension of category theory, that is, result from the first order theory of categories by adding finitary operations and equations. A category can be a restriction category or an allegory in different ways. (Note, however, that a category can be a Boolean restriction category in at most one way). For allegories, there are two new operations, intersection of relations and the unary $R^\circ = \{(y, x) : (x, y) \in R\}$ so that $R^\circ : Y \to X$ if $R : X \to Y$. I argue, now, that enlarging a restriction category to an allegory is often too big a jump. Consider the unique total function $f : \mathbb{N} \to 1$. Then $f^\circ : 1 \to \mathbb{N}$ maps one value to all natural numbers, and this seems far-fetched in a "typical" nondeterministic computational setting.

The tutorial talk discussed another reason to enlarge to a semiadditive category, namely to expand a network into a sum of paths. Here, it is not crucial that sum be always defined since only certain sums are needed, but this does allow a universal-algebraic description by building on abelian monoids as opposed to partial abelian monoids.

Challenge 1: Given a restriction category $\mathcal{X}$, find a "non-deterministic completion" $\widehat{\mathcal{X}}$ which is semiadditive and as "computationally viable" as $\mathcal{X}$ was.

Such $\widehat{\mathcal{X}}$ should at least be a support category (i.e. the fourth restriction axiom $\overline{g}\,f = f\overline{gf}$ which expresses that f is deterministic is weakened to the axiom of support that $\overline{\overline{g}\,f} = \overline{gf}$) in which $\mathcal{X}$ is embedded so as to preserve $\overline{f}$.

Now **Rel** has support $\overline{R} = 1 \cap R^\circ R = \{(x,x) : \exists\, y\ xRy\}$ and **Pfn** $\subset$ **Rel** preserves $\overline{f}$. The principal objection is lack of computational viability.

Notice that the partial order $R \leq S$ if $S\overline{R} = R$ is not subset inclusion but is rather the extension ordering of the restriction category **Pfn** thinking of a relation from X to Y as a partial function from X to the nonempty subsets of Y.

Let $\mathcal{X}$ be a locally small preadditive category. An **ideal** $I \subset \mathcal{X}(X,Y)$ satisfies $0 \in I$ and, for summable $f,g : X \to Y$, $f,g \in I \Leftrightarrow f+g \in I$. Every intersection of ideals again is, so let $I(A)$ be the ideal generated by $A \subset \mathcal{X}(X,Y)$. Let $\widehat{\mathcal{X}}(X,Y)$ be the set of all finitely generated ideals in $\mathcal{X}(X,Y)$. [25, Theorem 13.14] proves the following facts.

- $\widehat{\mathcal{X}}$ is a Boolean category (see the "footnote on Boolean categories" below), with composition well defined by $I(B) \circ I(A) = I(BA)$.
- $\mathcal{X} \to \widehat{\mathcal{X}}$, $f \mapsto I(f)$ is an embedding.
- If $P \xrightarrow{i} X \xleftarrow{i'} P'$ is a coproduct in $\mathcal{X}$ with projections $P \xleftarrow{\rho} X \xrightarrow{\rho'} P'$ then $P \xrightarrow{I(i)} X \xleftarrow{I(i')} P'$ is a coproduct in $\widehat{\mathcal{X}}$ and $P \xleftarrow{I(\rho)} X \xrightarrow{I(\rho')} P'$ is a product in $\widehat{\mathcal{X}}$. It follows that $\widehat{\mathcal{X}}$ is a semiadditive category. $I(A) + I(B) = I(A \cup B)$ so that the abelian monoid hom-sets are semilattices.
- Every Boolean functor $\mathcal{X} \to \mathcal{Y}$ with $\mathcal{Y}$ a Boolean semiadditive category whose hom-sets are semilattices uniquely extends to $\widehat{\mathcal{X}}$ as a Boolean functor.

Next observe that no restriction category can be its own completion:

Exercise 1A Show that if a semiadditive support category satisfies $\overline{0} = 0$ for all zero morphisms and is nontrivial in that some morphism is not 0, then the support cannot be a restriction. Hints. Apply the matrix calculus available in any semiadditive category. For $f,g : X \to Y$ consider $X \xrightarrow{(1\ \ 1)} X + X \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Y$. Show that $\overline{\begin{pmatrix} f \\ g \end{pmatrix}} = \begin{pmatrix} \overline{f} & 0 \\ 0 & \overline{g} \end{pmatrix}$. Show that the fourth restriction axiom for the maps above leads to the conclusion that $\overline{f} = \overline{f+g} = \overline{g}$.

Say that a relation $R : X \to Y$ is **bounded-valued** (bv) if there exists an integer n such that $\forall\, x\ |\{y : (x,y) \in R\}| \leq n$. The least such n is written $\|R\|$. Given $S : Y \to Z$ with R, S bv, SR is also bv and $\|SR\| \leq \|S\|\,\|R\|$.

Exercise 1B For a relation $R : X \to Y$ show that the following are equivalent.

1. R is bv.
2. R is a finite union of partial functions.
3. There exists a finitely generated ideal $I \subset \mathbf{Pfn}(X,Y)$ with $R = \bigcup I$.

(As **Pfn** is a Boolean restriction category, it is preadditive by Exercise 1, so ideals make sense). Hint. Show for any Boolean restriction category that $I(f_1,...,f_n) = \{g_1 \vee \cdots \vee g_n : g_i \perp g_j \text{ if } i \neq j,\ g_i \leq f_i\}$

By the second condition above, bv-relations are "computationally viable".

Exercise 1C Use the previous exercise to show that $\widehat{\mathbf{Pfn}}$ is the subcategory of **Rel** of all bv-relations, and is a support subcategory of **Rel**. Hints. One must prove $\bigcup I(A) = \bigcup I(B) \;\Rightarrow\; I(A) = I(B)$. To that end, write $I(B) = I(c_1, \ldots, c_n)$ with the c_i pairwise disjoint. For $a \in A$, partition its domain into the (possibly empty) blocks $X_1, \ldots, X_n$ with $X_i = \{x : ax = c_i x\}$ to conclude that a is a disjoint supremum of elements of $I(B)$.

We note that for $\mathcal{X}$ any Boolean restriction category, the support in $\widehat{\mathcal{X}}$ is well defined by $\overline{I(A)} = I(\{\overline{a} : a \in A\})$.

In any poset P with least element 0, say that $x \in P$ is a **Boolean element** if $\downarrow x = \{y : y \leq x\}$ is a Boolean algebra. Say that P is **Boolean generated** if every element is a finite supremum of Boolean elements.

Exercise 1D Show that $\widehat{\mathcal{X}}(X, Y)$ is a Boolean generated distributive lattice for any Boolean restriction category $\mathcal{X}$.

In any support category, say that a morphism $f : X \to Y$ is **deterministic** if for all $g : Y \to Z$, $\overline{g}\, f = f\, \overline{gf}$. The subcategory of deterministic morphisms is always a restriction category. For $\mathcal{X}$ a Boolean restriction category, all morphisms in $\mathcal{X}$ are deterministic in $\widehat{\mathcal{X}}$. When $\mathcal{X} = \mathbf{Pfn}$, all deterministic bv-relations are indeed partial functions.

Open Problem I For which Boolean restriction categories $\mathcal{X}$ does $\mathcal{X}$ coincide with the deterministic morphisms in $\widehat{\mathcal{X}}$?

Another semiadditive completion of **Pfn** is finite-valued relations. Unlike bv-relations, this is the Kleisli category of a submonad of the power set monad. Of course, the universal property of the ideal completion is lost.

Open Problem II Is there a general construction to complete a Boolean restriction category to a semiadditive Kleisli category for a monad on its total morphism category, which specializes to finite-valued relations when the category is **Pfn**?

An important source of examples of Boolean restriction categories is the partial morphism category of a Boolean topos, since such a topos is an extensive category in which every monic is a coproduct injection. The category of all relations over any topos is a tabular allegory with a number of nice properties [13].

Challenge 2: Toward a theory of restriction allegories.

We work in a split restriction category $\mathcal{X}$ with binary restriction products. If we can embed $\mathcal{X}$ in an allegory in which two relations have a union, a suitable subcategory could provide a useful semiadditive completion. The challenge is to define a *restriction allegory* with axioms (RA.1), (RA.2), ... of which is first is

(RA.1) Given a restriction idempotent $R : X \times Y \to X \times Y$ there exists a least $e : Y \to Y$ (in the restriction order) with

$$\begin{array}{ccc} X \times Y & \xrightarrow{\;R\;} & X \times Y \\ {\scriptstyle pr_Y}\downarrow & \geq & \downarrow{\scriptstyle pr_Y} \\ Y & \xrightarrow[\;e\;]{} & Y \end{array}$$

Note that e in (RA.1) is a restriction idempotent since $id_Y pr_Y \geq pr_Y R \Rightarrow e \leq id_Y$.

Definition A **relation** $X \rightharpoondown Y$ is a restriction idempotent $X \times Y \to X \times Y$. **Relation composition** $S \circ R : X \rightharpoondown Z$ given $R : X \rightharpoondown Y$, $S : Y \rightharpoondown Z$ is defined by (RA.1):

$$
\begin{array}{ccccc}
X \times Y \times Z & \xrightarrow{R \times 1} & X \times Y \times Z & \xrightarrow{1 \times S} & X \times Y \times Z \\
{\scriptstyle pr_{XZ}} \downarrow & & \geq & & \downarrow {\scriptstyle pr_{XZ}} \\
X \times Z & & \xrightarrow[S \circ R]{\qquad\qquad} & & X \times Z
\end{array}
$$

(noting that $R \times 1$, $1 \times S$, and hence their composition, are restriction idempotents).

One routinely checks that this gives the usual notion of relation and relation composition if $\mathcal{X}$ is the category **Pfn** of sets and partial functions.

In establishing the tabular allegory of a regular category, it is a mild task to establish the associativity of composition. Here we have:

Exercise 2A Show that relation composition is associative. Hint. Both $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are induced (via proper placement of parentheses in $W \times X \times Y \times Z$) by

$$
W \times X \times Y \times Z \xrightarrow{R \times 1 \times 1} W \times X \times Y \times Z \xrightarrow{1 \times S \times 1} W \times X \times Y \times Z \xrightarrow{1 \times 1 \times T} W \times X \times Y \times Z
$$

Definition The **opposite relation** $R^o : Y \rightharpoondown X$ of $R : X \rightharpoondown Y$ is the composition

$$
R^o \quad = \quad Y \times X \cong X \times Y \xrightarrow{R} X \times Y \cong Y \times X
$$

That R^o is a restriction idempotent is immediate from the next exercise.

Exercise 2B In any restriction category, given a commutative square

$$
\begin{array}{ccc}
W & \xrightarrow{p} & X \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle f} \\
Y & \xrightarrow[q]{} & Z
\end{array}
$$

with $p = \overline{p}$ and f total and epic, necessarily $q = \overline{q}$.

The restriction idempotents of an object form a meet semilattice. For relations we use the notations $R \subset S$ for the restriction ordering $RS = R$, and $R \cap S$ for the infimum RS.

Exercise 2C Establish the following allegory axioms:

- $R^{oo} = R$.
- $(S \circ R)^o = R^o S^o$.
- $(R \cap S)^o = R^o \cap S^o$.
- For $R : X \rightharpoondown Y$, $S, T : Y \rightharpoondown Z$, $S \subset T \Rightarrow S \circ R \subset T \circ R$.

Unaddressed so far is the *modular law* for $R : X \rightharpoondown Y$, $S : Y \rightharpoondown Z$, $T : X \rightharpoondown Z$,

$$(S \circ R) \cap T \ \subset\ S \circ (R \cap (S^o \circ T))$$

This might be true. We haven't had the fortitude to figure it out because of other aspects currently in limbo. Do we even have a category? How is $\mathcal{X}$ embedded in this category once we get one?

An obvious axiom to try is what [7] would possibly call the "axiom of discreteness", namely

(RA.2) Given $X \xleftarrow{i} P \xrightarrow{f} Y$ with i a restriction monic and f total, $[i\, f] : P \to X \times Y$ is again a restriction monic.

By the Cockett-Lack completeness theorem in [8], the partial morphism category of such $[i, f]$ is precisely $\mathcal{X}$ (noting that $\mathcal{X}$ is presumed split), so (RA.2) tells us how to embed morphisms in relations. This hopefully also provides identity relations. So far, I have not seen how to show without further axioms that partial morphism composition and relation composition coincide. Note, however, that all works correctly if $\mathcal{X} = \mathbf{Pfn}$.

Challenge 3: Determine when (RA.1, RA.2) force the split restriction category $\mathcal{X}$ with binary restriction products to be ranged.

Definition In a restriction category, say that $f : X \to Y$ is **restriction-surjective** if whenever $f = X \to Q \xrightarrow{j} Y$ with j a restriction monic, j is an isomorphism.

Exercise 3A Show that an epic restriction monic is an isomorphism. Conclude that every split epic is restriction-surjective.

It comes down to determining when the following axiom is true.

(RA.3) Given a pullback

$$\begin{array}{ccc} P & \xrightarrow{g} & Q \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f restriction-surjective and j a restriction monic, g is restriction surjective.

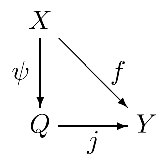
(RA.3) does not involve relations or restriction products and is conceivably true in any split restriction category. One would hope to adapt the proof of [6, Theorem 4.4] to prove this, but so far I haven't seen how. In the final corollary of this handout, however, we see that (RA.3) does hold for any Boolean restriction category.

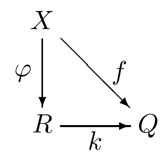
Exercise 3B Show that a restriction category in which the composition of two restriction-surjectives fails to be restriction-surjective leads to a counterexample to (RA.3). Hint. Consider the idempotent completion.

Theorem If (RA.1, RA.2, RA.3) hold, $\mathcal{X}$ is a ranged restriction category.

Proof By the theory of [6, Section 4.2] it suffices to show that every total morphism factors through a least restriction monic. Given total $f : X \to Y$ let $X \times Y \xrightarrow{s_f} X \xrightarrow{[1\, f]} X \times Y$ be the restriction idempotent corresponding to the restriction monic $[1\, f]$ and let $e_f : Y \to Y$ be the least morphism of (RA.1) corrresponding to $[1\, f]\, s_f$ with splitting $Y \xrightarrow{t} Q \xrightarrow{j} Y$. We will show that f factors through Q an that if f factors through the restriction monic subobject $k : R \to Y$

then $Q \subset R$. As f is total, $[1\,f]pr_Y = f$ so (RA.1) gives $jtpr_Y \geq fs_f$. Then $\overline{e_f pr_Y} \geq \overline{fs_f} = \overline{\overline{f}s_f} = \overline{s_f} = [1\,f]s_f$ so $[1\,f]\overline{e_f f} = [1\,f]\overline{e_f pr_Y[1\,f]} = \overline{e_f pr_Y}[1\,f] \geq [1\,f]s_f[1\,f] = [1\,f]$. It follows that $\overline{e_f f} = s_f[1\,f]\overline{e_f f} \geq s_f[1\,f] = id_X$ which gives $\overline{e_f f} = id_X$. But then $e_f f = \overline{e_f} f = f\overline{e_f f} = fid_X = f$. As $j = eq(e_f, i_X)$, we have unique ψ as in the left triangle below





(Indeed $\psi = X \xrightarrow{f} Y \xrightarrow{t} Q$). Such ψ is total as f is. Now suppose that $Q \xrightarrow{u} R \xrightarrow{k} Q$ splits a restriction idempotent of Q and that f factors through R so that φ exists as in the triangle on the right above. Then $id_R = R \xrightarrow{k} Q \xrightarrow{j} X \xrightarrow{t} Q \xrightarrow{u} R$ and $jkut$ is a restriction idempotent since $jkut = j\overline{u}t = jt\overline{ut} = \overline{jt}\,\overline{ut}$. We pause for

Exercise 3C For general posets, a monotone injective map need not reflect the order. Show, however, that if j is a monic in a restriction category that $jx \leq jy \Rightarrow x \leq y$.

By this exercise, $tpr_Y \geq \psi s_f$. We have

$$\varphi s_f = uk\varphi s_f = u\psi s_f = utpr_y\overline{\psi s_f} = utpr_Y\overline{k\varphi s_f} = utpr_Y\overline{\varphi s_f}$$

so $utpr_Y \geq \varphi s_f$ and $tpr_Y = kutpr_Y \geq k\psi s_f = \psi s_f$. As $jt = e_f$, $jt \leq jkut = j(ku)t \leq jt$ (as $ku = \overline{ku}$) so $jt = jkut$. As j is monic and t is epic, $ku = id_Q$. Thus k is split monic and epic, hence an isomorphism. Finally, let S be any restriction monic subobject through which f factors. Then f factors through $Q \cap S$. By the argument just given, $Q \cap S = Q$ so $Q \subset S$ as desired. $\square$

Footnote on Boolean categories Boolean categories were introduced in [25]. At that time, restriction categories had not yet appeared. We take this opportunity to advocate for the utility of these categories within the restriction framework.

Definition A **Boolean category** satisfies the following axioms.

(B.1) Finite coproducts exist.

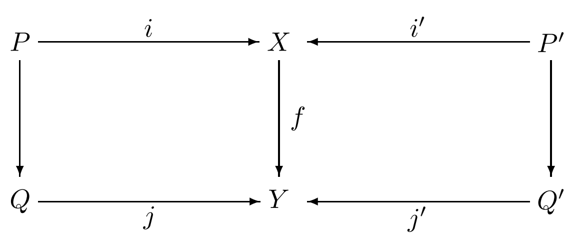
(B.2) The pullback of a coproduct injection along any morphism exists and is again a coproduct injection.

(B.3) A coproduct injection pulls back coproducts.

(B.4) If $X \xrightarrow{f} X \xleftarrow{f} X$ is a coproduct, $X = 0$ is the initial object.

In any Boolean category, coproduct injections are monic and $Summ(X)$, the poset of summands –those subobjects of X represented by a coproduct injection– is a Boolean algebra.

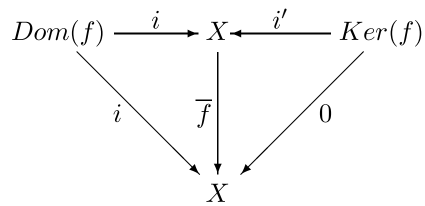
In any category, say that $f : X \to Y$ is **deterministic** if given a coproduct $Q \xrightarrow{j} Y \xleftarrow{j'} Q'$ there exists a coproduct $P \xrightarrow{i} X \xleftarrow{i'} P'$ and a commutative diagram



A morphism f in a category with an initial object 0 is **null** if it factors through 0 and is **total** if ft null $\Rightarrow t$ is null. In a Boolean category, the pullback of 0 along $f : X \to Y$ is the **kernel** $Ker(f)$ of f and its Boolean complement in $Summ(X)$ is the **domain** $Dom(f)$ of f. Here are some basic results.

Theorem [25, 27] For a category $\mathcal{X}$, the following are equivalent.

1. $\mathcal{X}$ is a Boolean restriction category.
2. $\mathcal{X}$ is a Boolean category for which 0 is a zero object and in which every map is deterministic.
3. $\mathcal{X}$ is a Boolean category for which 0 is a zero object and is such that $\overline{f}$ defined for $f : X \to Y$ by



   is a restriction.

The restrictions in (1, 3) are the same and the total maps are indeed those with $\overline{f} = 1$.

The next result is immediate from [25, Corollary 12.3].

Theorem A category is extensive if and only if it is a Boolean category in which all morphisms are total and deterministic.

It is then quite easy to show that the Cockett-Lack completeness theorem for restriction categories gives

Theorem A category is a Boolean restriction category if and only if it is isomorhic to a partial morphism category $Par(\mathcal{X}, \mathcal{M})$ with $\mathcal{X}$ an extensive category and $\mathcal{M}$ its class of coproduct injections.

An important tool for the study of any Boolean category $\mathcal{B}$ is its **Kozen functor** $K : \mathcal{B} \to$ **Rel** defined by

$$\begin{aligned} KX &= \{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter on the Boolean algebra } Summ(X)\} \\ (\mathcal{U}, \mathcal{V}) \in Kf &\Leftrightarrow V \in \mathcal{V} \Rightarrow <f> V \in \mathcal{U} \end{aligned}$$

where $<f> V = ([f]V')'$ and $[f]Q$ is the pullback. This functor was introduced in [25, Section 11], being adapted from Dexter Kozen's work [21, 3.8'] on dynamic algebras. K has the following properties:

- K preserves $X + Y$ and $[f]Q$.
- K preserves and reflects 0.
- K induces a boolean algebra injection $Summ(X) \to 2^{KX}$.
- K preserves and reflects null morphisms.
- K preserves and reflects total morphisms.

Note that, in general, K need not be faithful. Sets and bag-valued functions provides an example, K being the forgetful to **Rel**. Say that a map f in a category is **summand-surjective** if it factors through no proper summand of its codomain. In **Rel**, this is just onto, that is, every element of the codomain is related to at least one element of the domain. The next result didn't quite make it into [25] so is observed here.

Proposition The following hold.

- K preserves and reflects summand-surjectives.
- K preserves and reflects deterministic morphisms.

Proof for the first statement, "preserves" is [25, Theorem 11.23], whereas "reflects" is immediate since if f factors $X \to Q \to Y$ with Q a summand and Kf onto, $KQ \to KY$ is onto so that $Q = Y$. We turn to the second statement. For "preserves", suppose $f : X \to Y$ with Kf not a partial function, so that there exists $(\mathcal{U}, \mathcal{V})$, $(\mathcal{U}, \mathcal{W}) \in Kf$ with $\mathcal{V} \neq \mathcal{W}$. By [25, Proposition 12.2], a morphism f in a Boolean category is deterministic $\Leftrightarrow <\!f\!> (Q \cap Q') = <\!f\!> Q \cap <\!f\!> Q'$ $\Leftrightarrow$ $[f](Q \cup R) = [f]Q \cup [f]R$. Thus if f is deterministic, choose a summand V with $V \in \mathcal{B}$, $V' \in \mathcal{W}$ and observe

$$0 = <\!f\!> 0 = <\!f\!> (V \cap V') = <\!f\!> V \cap <\!f\!> V' \in \mathcal{U}$$

a contradiction. For the converse, suppose Kf is a partial function. Then in 2^{KX}, $K([f]Q \cup [f]R) = [Kf]Q \cup [Kf]R = [Kf](Q \cup R) = K([f](Q \cup R))$ so that $[f]Q \cup [f]R = [f](Q \cup R)$ since $KX \to 2^{KX}$ is injective. □

Notice that, by the above, the Kozen functor of a Boolean restriction category maps into **Pfn**.

The following illustrates the powerful metatheoretic consequences of the Kozen functor.

Corollary In any Boolean category $\mathcal{B}$, given a pullback

$$\begin{array}{ccc} P & \xrightarrow{g} & Q \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f deterministic and summand-surjective and Q a summand, g is summand-surjective.

Proof In **Rel**, the pullback is $P = \{x \in X : (x, y) \in f \Rightarrow y \in Q\}$. If f is onto and $q \in Q$, there exists x with $(x, q) \in f$. As f is a partial function such q is unique, so $x \in P$. This shows g is onto as well. Back in $\mathcal{B}$, the argument shows that Kg is onto, so g is summand-surjective. □

In a Boolean restriction category, the restriction monics are the coproduct injections, so the restriction-surjectives are just the summand-surjectives. We then immediately have from the theorem of Challenge 3 that

Corollary A Boolean restriction category with restriction products satisfying (RA.1), (RA.2) is a ranged restriction category.

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