(for slide 4)

(R.1) $f\,\overline{f} = f \qquad X \xrightarrow{f} Y$

(R.2) $\overline{f}\,\overline{g} = \overline{g}\,\overline{f} \qquad Y \xleftarrow{f} X \xrightarrow{g} Z$

(R.3) $\overline{g\,\overline{f}} = \overline{g}\,\overline{f} \qquad Y \xleftarrow{f} X \xrightarrow{g} Z$

(R.4) $\overline{g}\,f = f\,\overline{gf} \qquad X \xrightarrow{f} Y \xrightarrow{g} Z$

Consequences

(A) $\overline{f}\,\overline{f} = \overline{f}$

(B) $\overline{f}\,\overline{gf} = \overline{gf}$

(C) $\overline{\overline{f}} = \overline{f}$

(D) $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}$

(E) $\overline{\overline{g}\,f} = \overline{gf}$

(for slide 19)

Let a be regular, so that $\exists x$ with $axa = a$. Then

$$a(xax)a = ax(axa) = axa = a$$

whereas

$$(xax)a(xax) = x(axa)xax = xaxax = xax$$

Thus a a has xax as inverse.

(for slide 31)

We want $\lambda_a = \lambda_{\overline{a}}$.

Let $\lambda_a\, x = ax$ with $x \in D_a$ with D_a yet to be defined, undefined else.

Thus $D_a = D_{\overline{a}}$ and $x \in D_a \Rightarrow \overline{a}\, x = x$.

So define $D_a = \{x : \overline{a}\, x = x\}$. Indeed $D_a = D_{\overline{a}}$ because $\overline{\overline{a}} = \overline{a}$.

The domain of $\lambda_a \lambda_b$ is x with $\overline{b}x = x, \overline{a}bx = bx$. The domain of λ_{ab} is x with $\overline{ab}\, x = x$.

If $\lambda_a \lambda_b\, x$ is defined then $\overline{ab}x = x\, \overline{abx}(R.4) = x\, \overline{\overline{a}\, bx} = x\, \overline{bx} = \overline{b}\, x(R.4) = x$.

If $\lambda_{ab}\, x$ is defined then $\overline{a}\, bx = b\, \overline{ab}\, x(R.4) = bx$ and $\overline{b}\, x = \overline{b}\, \overline{ab}\, x = \overline{ab}\, x = x$.

Thus λ is a restriction algebra homomorphism. If $\lambda_a = \lambda_b$ then $\overline{a}x = x \Leftrightarrow \overline{b}x = x$ and then $ax = bx$. As $\overline{a}\, \overline{a} = \overline{a}$, $\overline{b}\, \overline{a} = \overline{a}$ so $\overline{a} \leq \overline{b}$. By symmetry, $\overline{a} = \overline{b}$. Thus

$$a = a\, \overline{a} = a\, \overline{b} = b\, \overline{b} = b$$

(for slide 51)

Let S be left normal. In particular, S is normal, so is a strong semilattice of rectangular bands S_e. Each S_e is a subsemigroup of a left normal semigroup, so is a left normal— as well as a rectangular band. Thus, in S_e,

$$xy = xxy = xyx = x$$

Conversely, if S is a strong semilattice of left zeroes Fe then axy has form $\alpha\beta\gamma$ in $F(axy)$ whereas ayx is then $\alpha\gamma\beta$ in $F(axy)$ and both are α because $F(axy)$ is a left zero.

(for slide 52)

No reliance on the Yamada–Kimura theorem.

For S a band with restriction,

$$\overline{xy} = \overline{x\,\overline{y}} = \overline{x}\,\overline{y}$$

Let $Fe = \{x : \overline{x} = e\}$. For $x, y \in Fe$, $xy = x\,\overline{y} = xe = x\,\overline{x} = x$ which shows that Fe is a left zero semigroup.

Let $e \geq f$. Define $F_{ef} : Fe \to Ff$ by $x \mapsto xf$. Then $\overline{xf} = \overline{x}\,\overline{f} = ef = f$ shows that this is well defined.

By left zero, all functions are homomorphisms.

For $x \in Fe$, $y \in Ff$,

$$F_{e,ef}(x)\,F_{f,ef}(y) = F_{e,ef}(x) \text{ (left zero) } = xef = x\overline{x}f = xf = x\overline{y} = xy$$

$R(x) \subset S \xrightarrow{\psi} R(S) = id$, $\psi(x) = \overline{x}$.

Converse: Exercise