Unwirings and exponentiability for multicategories

Nathan Bowler

October 24, 2010

The intuition for multicategories

There are many different species of multicategory. A multicategory of a given species is like a category, but with the following tweaks (the exact details of which will depend on the species):

- The sources of the arrows are not just objects, they are combinations of objects (the manner of combination depends on the species of multicategory). The targets are still just objects.
- The classes of objects and arrows may have extra structure, which will also depend on the species.

The definition of multicategories

A species of multicategory is determined by a weak double category $\mathcal{E}$ and a monad T on that weak double category. An $(\mathcal{E}, T)$-multicategory is given by

- a 0-cell a_0 of $\mathcal{E}$.
- a horizontal 1-cell $a_0 \xrightarrow{a_1} Ta_0$ of $\mathcal{E}$.
- 2-cells

$$\begin{tikzcd}
a_0 \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow\mathrm{ids}"] & a_0 \arrow[d, "\eta_{a_0}"] \\
a_0 \arrow[r, "a_1"'] & Ta_0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[drr, phantom, "\Downarrow\mathrm{comp}"] & Ta_0 \arrow[r, "Ta_1"] & T^2a_0 \arrow[d, "\mu_{a_0}"] \\
a_0 \arrow[rr, "a_1"'] & & Ta_0
\end{tikzcd}$$

satisfying certain conditions, called the associativity and identity laws.

The associativity and identity laws

Identity law:

$$\begin{tikzcd}
a_0 \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow\mathrm{ids}"] & a_0 \arrow[r, "a_1"] \arrow[d, "\eta_{a_0}"] \arrow[dr, phantom, "\Downarrow\eta_{a_1}"] & Ta_0 \arrow[d, "\eta_{Ta_0}"] \\
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[drr, phantom, "\Downarrow\mathrm{comp}"] & Ta_0 \arrow[r, "Ta_1"] & T^2a_0 \arrow[d, "\mu_{a_0}"] \\
a_0 \arrow[rr, "a_1"'] & & Ta_0
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[d, phantom, shift left=4ex, "\Downarrow 1"] & Ta_0 \arrow[d, "1"] \\
a_0 \arrow[r, "a_1"'] & Ta_0
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow 1"] & Ta_0 \arrow[r, equal] \arrow[d, "1"] \arrow[dr, phantom, "\Downarrow T\,\mathrm{ids}"] & Ta_0 \arrow[d, "T\eta_{a_0}"] \\
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[drr, phantom, "\Downarrow\mathrm{comp}"] & Ta_0 \arrow[r, "Ta_1"] & T^2a_0 \arrow[d, "\mu_{a_0}"] \\
a_0 \arrow[rr, "a_1"'] & & Ta_0
\end{tikzcd}$$

Associativity law:

$$\begin{tikzcd}
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[drr, phantom, "\Downarrow\mathrm{comp}"] & Ta_0 \arrow[r, "Ta_1"] & T^2a_0 \arrow[r, "T^2a_1"] \arrow[d, "\mu_{a_0}"] \arrow[dr, phantom, "\Downarrow\mu_{a_1}"] & T^3a_1 \arrow[d, "\mu_{Ta_0}"] \\
a_0 \arrow[rr, "a_1"] \arrow[d, "1"'] \arrow[drrr, phantom, "\Downarrow\mathrm{comp}"] & & Ta_0 \arrow[r, "Ta_1"] & T^2a_0 \arrow[d, "\mu_{a_0}"] \\
a_0 \arrow[rrr, "a_1"'] & & & Ta_0
\end{tikzcd}
=
\begin{tikzcd}
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow 1"] & Ta_0 \arrow[r, "Ta_1"] \arrow[d, "1"] \arrow[drr, phantom, "\Downarrow T\,\mathrm{comp}"] & T^2a_0 \arrow[r, "T^2a_1"] & T^3a_0 \arrow[d, "T\mu_{a_0}"] \\
a_0 \arrow[r, "a_1"] \arrow[d, "1"'] \arrow[drrr, phantom, "\Downarrow\mathrm{comp}"] & Ta_0 \arrow[rr, "Ta_1"] & & T^2a_0 \arrow[d, "\mu_{a_0}"] \\
a_0 \arrow[rrr, "a_1"'] & & & Ta_0
\end{tikzcd}$$

A few species of multicategory

$\mathcal{E}$	T	$(\mathcal{E}, T)$-multicategories
Span(**Set**)	identity	categories
Span(**Set**)	list	plain multicategories
Span(**Gph**)	path	virtual double categories
Span($\mathcal{C}/C_0$)	T/C	(**Span**($\mathcal{C}$), T)-multicategories over C
Rel	ultrafilter	topological spaces

Unwirable maps of algebras and unwirings

Let T be a cartesian monad on a cartesian category $\mathcal{C}$. A map $f\colon (A, \alpha) \to (B, \beta)$ of T-algebras is unwirable iff

$$\begin{tikzcd} TB \arrow[r, "Tf"] \arrow[d, "\beta"'] & TA \arrow[d, "\alpha"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback.

Dropping the requirement that B have a T-algebra structure, an unwiring of (B, f) is a map $\nu\colon B \times_A TA \to TB$ making the following diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "B\times_A \eta_A"] \arrow[d, "\eta_B"'] & B \times_A TA \arrow[d, "\pi'"] \arrow[dl, "\nu"'] \\ TB \arrow[r, "Tf"'] & TA \end{tikzcd} \qquad \begin{tikzcd} B \times_A T^2A \arrow[r, "\nu_{TA}"] \arrow[d, "B\times_A \mu_A"'] & T(B \times_A TA) \arrow[r, "T\nu"] & T^2B \arrow[d, "\mu_B"] \\ B \times_A TA \arrow[rr, "\nu"'] & & TB \end{tikzcd}$$

Unwirings which are isomorphisms correspond to unwirable maps of T-algebras.

Unwirings as distributive laws

For any object $C \xrightarrow{g} A$ of $\mathcal{C}/A$, pulling back ν along $T\pi\colon T(B \times_A C) \to TB$ gives a diagram

$$\begin{tikzcd} B \times_A TC \arrow[r, "I(\nu)_g"] \arrow[d, "B\times_A Tg"'] & T(B \times_A C) \arrow[r, "T\pi'"] \arrow[d, "T\pi"] & TC \arrow[d, "Tg"] \\ B \times_A TA \arrow[r, "\nu"'] & TB \arrow[r, "Tf"'] & TA\,. \end{tikzcd}$$

Since the maps $I(\nu)_g$ are formed in this way by pullback, they collectively form a cartesian natural transformation $I(\nu)\colon B \times_A T- \to T(B \times_A -)$.

For any unwiring ν of B as above, $I(\nu)$ is a distributive law of the comonad $B \times_A -$ over T/A. This gives a correspondence between unwirings ν of B and cartesian distributive laws of $B \times_A -$ over T/A.

Unwirings as exponentiability-lifters

For any unwiring ν of B, $I(\nu)$ gives $B \times_A -$ the structure of a cartesian colax map of monads from T/A to itself. Suppose that f is exponentiable as a map in $\mathcal{C}$. Let $m(\nu)$ be the mate of $I(\nu)$ with respect to the adjuction $B \times_A - \dashv (-^B)_A$. Then $((-^B)_A, m(\nu))$ is a lax map of monads from (T/A) to itself. Thus the functor $(-^B)_A$ lifts to an endomorphism of the category T-**Alg**$/(A, \alpha)$ of (T/A)-algebras. In particular, any unwirable map of T-algebras is exponentiable.

Unwirability for multicategories in the sense of Leinster

Now let T be a suitable monad (in the sense of Leinster) on a cartesian category $\mathcal{C}$. So we have a 'free T-multicategory' monad T^+ on $\mathcal{C}$-**Gph**. A map f of T-multicategories is unwirable iff the squares

$$\begin{tikzcd} B_0 \arrow[r, "\text{ids}"] \arrow[d, "f_0"'] & B_1 \arrow[d, "f_1"] \\ A_0 \arrow[r, "\text{ids}"'] & A_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_1 \circ B_1 \arrow[r, "\text{comp}"] \arrow[d, "f_1 * f_1"'] & B_1 \arrow[d, "f_1"] \\ A_1 \circ A_1 \arrow[r, "\text{comp}"'] & A_1 \end{tikzcd}$$

are both pullbacks. In fact, if the second of these squares is a pullback then the first must also be.
A T-multicategory is unwirable iff the unique map ! from it to the terminal T-multicategory is unwirable.

Cartesian 2-cells in double categories

Suppose that in some double category we have a 2-cell

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & B' \arrow[d, "g'"] \\ C \arrow[r, "n"'] & C' . \end{tikzcd}$$

We say θ is *cartesian* iff for any $f\colon A \to B$ and $f'\colon A' \to B'$, any other 2-cell

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, "g \cdot f"'] \arrow[dr, phantom, "\Downarrow\phi"] & A' \arrow[d, "g' \cdot f'"] \\ C \arrow[r, "n"'] & C' \end{tikzcd} \quad \text{factors through } \theta \text{ uniquely as} \quad \begin{tikzcd} A \arrow[r, "l"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\widehat{\phi}"] & A' \arrow[d, "f'"] \\ B \arrow[r, "m"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & B' \arrow[d, "g'"] \\ C \arrow[r, "n"'] & C' . \end{tikzcd}$$

Cartesian cells in **Span**(**Set**)

For example, a 2-cell

$$\begin{tikzcd} & r \arrow[dl, "p"'] \arrow[d, "h"] \arrow[dr, "q"] & \\ b \arrow[d, "g"'] & u \arrow[dl, "s"'] \arrow[dr, "t"] & b' \arrow[d, "g'"] \\ c & & c' \end{tikzcd}$$

in **Span**(**Set**) is cartesian iff r is the limit of the diagram

$$\begin{tikzcd} b \arrow[d, "g"'] & u \arrow[dl, "s"'] \arrow[dr, "t"] & b' \arrow[d, "g'"] \\ c & & c' \end{tikzcd}$$

with p, h and q being legs of the limit cone

Translating from Leinster's world to Cruttwell and Shulman's

The unit map of the terminal T-multicategory is the component of η at 1, and so in this case the first pullback square in the definition of an unwirable T-multicategory factors as

$$\begin{tikzcd} a_0 \arrow[r, "1"] \arrow[d, "\text{ids}"'] & a_0 \arrow[r, "!"] \arrow[d, "\eta_{a_0}"] & 1 \arrow[d, "\eta_1"] \\ a_1 \arrow[r, "s"'] & Ta_0 \arrow[r, "T!"'] & T1 . \end{tikzcd}$$

The right hand square is a pullback, so the whole thing is a pullback iff the left hand square is, that is iff

$$\begin{tikzcd} & a_0 \arrow[dl, "1"'] \arrow[d, "\text{ids}"] \arrow[dr, "1"] & \\ a_0 \arrow[d, "1"'] & a_1 \arrow[dl, "t"'] \arrow[dr, "s"] & a_0 \arrow[d, "\eta_{a_0}"] \\ a_0 & & Ta_0 \end{tikzcd} \quad \text{is cartesian.}$$

Normalised and unwirable multicategories

Definition
A multicategory is *normalised* iff the identity 2-cell is cartesian.

Definition
A normalised multicategory is *unwirable* iff the composition 2-cell is also cartesian.

A correspondence between kinds of T-algebra in the horizontal bicategory and these structures

lax	multicategories
weak	unwirable multicategories
colax	unwirings

Normalised and unwirable topological spaces

A topological space is normalised iff it is T_1.

A T_1 topological space X is unwirable iff for any point x in X and any open neighbourhood U of x there is another open neighbourhood V of x such that any open cover of U has a finite subset covering V. Such a topological space is called *quasi locally compact*. The *quasi locally compact* spaces are precisely the exponentiable objects of **Top**.

Conjecture
For sufficiently friendly species of multicategory, amongst the normalised multicategories it is precisely the unwirable ones which are exponentiable.

This is more useful than it appears because

Theorem
For every species of multicategory we can construct another species so that the multicategories of the first species are exactly the normalised multicategories of the second.

References

Nathan Bowler.
A unified approach to the construction of categories of games.
PhD thesis, University of Cambridge, 2010.

G. S. H. Cruttwell and Michael A. Shulman.
A unified framework for generalized multicategories, 2009.

Tom Leinster.
Higher Operads, Higher Categories.
Number 298 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.