Differential and tangent structure for restriction categories

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(joint work with Robin Cockett and Jonathan Gallagher)

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Differential Restriction Categories

Recall that a **differential restriction category** is a cartesian left additive restriction category with a differentiation operation

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D[f]]{} Y}$$

such that

[DR.1] $D[f+g] = D[f] + D[g]$ and $D[0] = 0$;

[DR.2] $\langle g+h, k\rangle D[f] = \langle g, k\rangle D[f] + \langle h, k\rangle D[f]$ and $\langle 0, g\rangle D[f] = \overline{gf}\,0$;

[DR.3] $D[\pi_0] = \pi_0\pi_0$, and $D[\pi_1] = \pi_0\pi_1$;

[DR.4] $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$;

[DR.5] $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$;

[DR.6] $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \overline{h}\langle g, k\rangle D[f]$;

[DR.7] $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$;

[DR.8] $D[\overline{f}] = (1 \times \overline{f})\pi_0 = \overline{\pi_1 f}\,\pi_0$;

[DR.9] $\overline{D[f]} = 1 \times \overline{f} = \overline{\pi_1 f}$.

Examples

Some examples:

- smooth maps defined on open subsets of $\mathcal{R}^n$;
- rational functions over a commutative ring;
- the join completion of a differential restriction category;
- the classical completion of a differential restriction category (adds germs of maps).

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As compared to synthetic differential geometry, this is a very weak setting in which to talk about differentiation: we only assume finite products, no closed structure, no subobject classifier, no object of infinitesimals, no negatives.

Smooth manifolds?

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In particular, the derivative (or "push-forward") of a smooth map $M \xrightarrow{f} N$ between smooth manifolds is not a map

$$M \times M \xrightarrow{Df} N$$

but instead a map

$$TM \xrightarrow{DF} TN$$

where TM is the *tangent bundle* of M - think of it as consisting of a point on M, together with a tangent vector at that point.

Manifolds

To understand the problem, we first need to understand "manifolds" in general.

Definition

(Grandis/Cockett) If $\mathbf{X}$ is a restriction category, a manifold in $\mathbf{X}$ consists of a family of objects $U_i \in \mathbf{X}$, together with a family of maps $U_i \xrightarrow{\phi_{ij}} U_j$ such that:

1. $\phi_{ii}\phi_{ij} = \phi_{ij}$;
2. $\phi_{ij}\phi_{jk} \leq \phi_{ik}$;
3. ϕ_{ij} has partial inverse ϕ_{ji}; that is, $\phi_{ij}\phi_{ji} = \overline{\phi_{ij}}$.

The maps ϕ_{ij} define how the charts U_i overlap.

Category of Manifolds

There is a natural notion of morphism of these:

Definition

If (U_i, ϕ_{ij}) and (V_k, ψ_{kh}) are manifolds in $\mathbf{X}$, then a manifold morphism A consists of a family of maps $U_i \xrightarrow{A_{ik}} V_k$ such that:

1. $\phi_{ii} A_{ik} = A_{ik}$;
2. $\phi_{ij} A_{jk} \leq A_{ik}$;
3. $A_{ik} \psi_{kh} = \overline{A_{ik}}\, A_{ih}$

The last condition ensures the maps glue together correctly on the overlap of charts.

Examples

If **X** has joins, there is a restriction category **Mf**(**X**), where composition is given by matrix multiplication. (In fact, **Mf** is a 2-functor). Some examples:

- real manifolds (with coproducts);
- manifolds with corner, manifolds with boundary;
- simplicial complexes;
- schemes (?).

There are probably many other examples.

The Problem, re-stated:

The manifold completion of a differential restriction category is *not* a differential restriction category.

Tangent Bundles

So what can we do? Well, the derivative should use the tangent bundle, so we change our question: can we define the tangent bundle of any object in the manifold completion of a differential restriction category?

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Definition

Suppose that $\mathbf{X}$ is a differential restriction category, and $M = (U_i, \phi_{ij}) \in \mathbf{Mf}(\mathbf{X})$. Define TM to have:

- the same index set as M;
- charts $U_i \times U_i$ (locally, the tangent bundle is a product);
- transition maps

$$U_i \times U_i \xrightarrow{\langle D\phi_{ij}, \pi_1 \phi_{ij} \rangle} U_j \times U_j.$$

The tangent bundle is a manifold

Of course, we now need to check that this is a manifold. For example, to check the first manifold axiom, consider

$$\langle D\phi_{ii}, \pi_1\phi_{ii}\rangle\langle D\phi_{ij}, \pi_1\phi_{ij}\rangle$$

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as required. The other two axioms also use the chain rule, as well as DR8 and DR9.

T as a functor

One can similarly define an action on T of manifold maps:

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Moreoever, there is a natural transformation $TM \xrightarrow{p_M} M$ given by projection

$$U_i \times U_i \xrightarrow{\pi_1 \phi_{ij}} U_j$$

All of this only uses DR5, DR8, and DR9. What do the other differential axioms give us?

Adding tangent vectors

We should be able to add tangent vectors in the same tangent space. Since we can't talk about tangent spaces directly, we first have to show that the pullback of two copies of $TM \xrightarrow{p_M} M$ exists:

$$\begin{array}{ccc} T_2M & \xrightarrow{p_2} & TM \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ TM & \xrightarrow[p]{} & M \end{array}$$

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(think of the elements of T_2M as consisting of a pair of tangent vectors at a point). We can then use DR2 to show that we have natural transformations

$$T_2M \xrightarrow{+_M} TM \text{ and } M \xrightarrow{0_M} TM$$

that give an associative, unital, commutative addition.

Other results

We can use the other DR axioms to get other important properties of the tangent bundle:

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- using DR7, we get a natural transformation $T^2M \xrightarrow{c_M} T^2M$, which is known in DG as the "canonical flip";
- using DR1, we can get a sense in which addition is preserved, in that the following diagram commutes:

$$\begin{array}{ccc} TT_2M & \xrightarrow{\langle T(p_1)c, T(p_2)c \rangle} & T_2TM \\ {\scriptstyle T(+)}\downarrow & & \downarrow{\scriptstyle +_T} \\ T^2M & \xrightarrow[c]{} & T^2M \end{array}$$

Axiomatizing all this, we get the following:

Definition: Tangent Structure

If **X** is a cartesian restriction category, then **tangent structure** on **X** consists of:

- **(tangent bundles)** a cartesian restriction functor $\mathbf{X} \xrightarrow{T} \mathbf{X}$;
- a total natural transformation $T \xrightarrow{p} I$;
- for each n and M, the (restriction) pullback of n copies of $TM \xrightarrow{p_M} M$ exists (denote it by T_n);
- **(addition of tangent vectors**) total natural transformations $T_2 \xrightarrow{+} T$ and $I \xrightarrow{0} T$ which are unital, associative, and commutative;
- **(vertical lift)** a total natural transformation $T_2 \xrightarrow{l} T^2$;
- **(canonical flip)** a total natural transformation $T^2 \xrightarrow{c} T^2$;

together with some coherence equations. If **X** has joins, we also ask that T preserves them.

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Moreover, if **X** is a differential restriction category, there is tangent structure on **X**, where $TX = X \times X$ and

$$Tf = \langle Df, \pi_1 f \rangle$$

(in fact, such tangent structure is equivalent to giving differential structure!).

Thus, the fact that **Mf**(**X**) has tangent structure can be seen much more directly: we simply apply **Mf** to the tangent structure on **X**.

Conclusion

Some conclusions:

- tangent structure occurs whenever you have differential structure;
- tangent structure is preserved by well-behaved constructions, while differential structure is not;
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Some conclusions:

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Some questions:

- To what extent are the listed properties the essential properties of the tangent bundle? What else do they give?
- What do tangent structures look like in another settings (eg., classical completion, rational functions)?
- What other assumptions on $\mathbf{X}$ do we need to make to get more of differential geometry?