preamble

Differential Join Restriction Categories

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- Talk Outline

Background

Differential Restriction Categories

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Talk Outline

Goal: Give and motivate the definition of differential join restriction category.

Here are the ideas outlining the talk.

- Restriction categories axiomatize partiality.
- Cartesian differential categories axiomatize smooth functions on $\mathbb{R}^n$.
- Differential restriction categories combine the two theories to axiomatize the category of smooth functions on an open subset of $\mathbb{R}^n$.
- Differential join restriction categories bring more topological structure.

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Restriction Categories

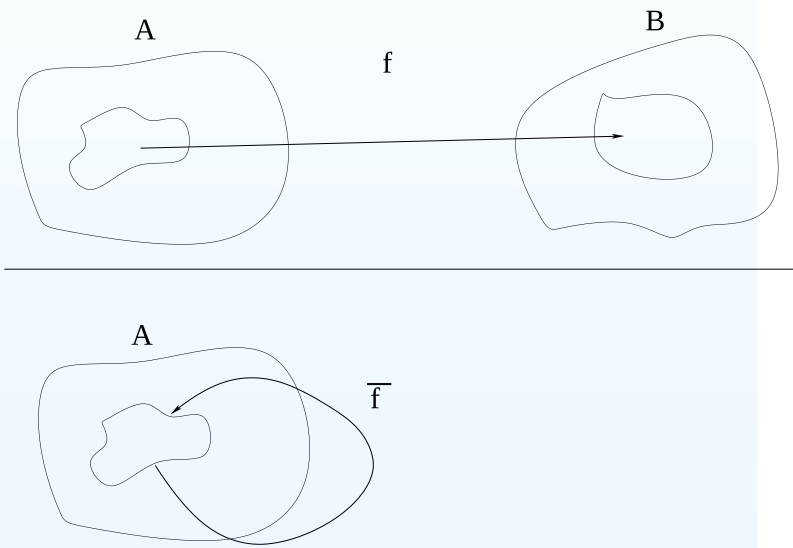
Definition 1. *A* **restriction category** *is a category* $\mathbb{X}$ *with a combinator,* $\overline{(\ )} : \mathbb{X}(A,B) \to \mathbb{X}(A,A)$*, satisfying*

R.1 $\overline{f}\, f = f$;

R.2 $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$;

R.3 $\overline{f}\,\overline{g} = \overline{\overline{f}\, g}$;

R.4 $f\,\overline{h} = \overline{f h}\, f$.



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Restriction Categories Examples

- PAR the category of sets and partial functions is a restriction category. $\overline{f}$ gives the domain of definition of f.

$$\overline{f}(x) = \begin{cases} x & f(x)\downarrow \\ \uparrow & \text{else} \end{cases}$$

- TOP the category of topological spaces and continous maps defined on an open set is a restriction category. This category has the same restriction as PAR.

Other examples of restriction categories can be found in [2]

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Cartesian Differential Categories

Cartesian Differential Categories [1] axiomatize smooth functions on $\mathbb{R}^n$ by axiomatizing a differential combinator (think Jacobian). The differential combinator has the type,

$$\frac{f : \mathbb{R}^n \to \mathbb{R}^m}{D[f] : \mathbb{R}^n \to (\mathbb{R}^n \multimap \mathbb{R}^m)}$$

It is too strong to assume that the category is closed with respect to linear maps. Thus the differential combinator is used in uncurried form.

$$\frac{f : \mathbb{R}^n \to \mathbb{R}^m}{D[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m}$$

The first coordinate is the directional vector. The second coordinate is the point of differentiation. This axiomatization will require products. Left additivity is needed for vectors.

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To build the theory of differential restriction categories, change the theory of cartesian differential categories in light of restriction structure. This means reconsidering:

- Cartesian categories,
- Left additive categories and cartesian left categories, and
- Differential categories.

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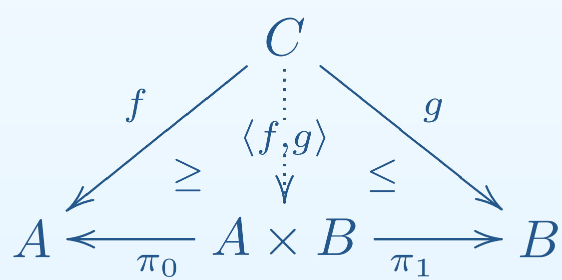
Pairing two maps together in a restriction category brings up the partiality of both.

Definition 2. *A map in a restriction category is* **total** *when* $\overline{f} = 1$.

Definition 3. *A* **restriction product** *of* A, B *is an object* $A \times B$ *such that for any* $f : C \to A, g : C \to B$ *there is a unique* $\langle f, g \rangle : C \to A \times B$ *such that*



$$a \leq b \Leftrightarrow \overline{a}\, b = a$$

where π_0, π_1 *are total and* $\overline{\langle f, g \rangle} = \overline{f}\,\overline{g}$.
A **restriction terminal object** *is* $\mathbf{1}$ *such that for any object* A*, there is a unique total map* $!_A : A \longrightarrow \mathbf{1}$ *which satisfies* $!_{\mathbf{1}} = id_{\mathbf{1}}$*. Further, for any map* $f : A \longrightarrow B$*,* $f!_B \leq !_A$*.*
A **cartesian restriction category** *has all restriction products.*

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The addition of two maps must only be defined when both are.

Definition 4. *A* **left additive restriction category** *has each* $\mathbb{X}(A, B)$ *a commutative monoid with* $\overline{f + g} = \overline{f}\,\overline{g}$ *and* 0 *being total. Furthermore,* $h(f + g) = hf + hg$ *and* $s0 = \overline{s}\,0$

Definition 5. *A map,* h*, in a left additive restriction category is* **total additive** *if* h *is total, and* $(f + g)h = fh + gh$*.*

Definition 6. *A* **cartesian left additive restriction category** *is both a left additive restriction category and a cartesian restriction category where* $\pi_0, \pi_1,$ *and* Δ *are total additive, and* $(f + h) \times (g + k) = (f \times g) + (h \times k)$*.*

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A **differential restriction category** is a cartesian left additive restriction category with a differential combinator

$$\frac{f : X \to Y}{D[f] : X \times X \to Y}$$

such that

DR.1 $D[f+g] = D[f] + D[g]$ and $D[0] = 0$ **(additivity of the differential combinator)**;

DR.2 $\langle g+h, k\rangle D[f] = \langle g, k\rangle D[f] + \langle h, k\rangle D[f]$ and $\langle 0, g\rangle D[f] = \overline{g} f 0$ **(additivity of differential in first coordinate)**;

DR.3 $D[1] = \pi_0$, $D[\pi_0] = \pi_0\pi_0$, and $D[\pi_1] = \pi_0\pi_1$;

DR.4 $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$;

DR.5 $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$ **(Chain rule)**;

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$$\frac{f : X \to Y}{D[f] : X \times X \to Y}$$

(... and)

DR.6 $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \overline{h}\langle g, k\rangle D[f]$ **(linearity of the derivative)**

DR.7 $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$ **(independence of partial derivatives)**;

DR.8 $D[\overline{f}] = (1 \times \overline{f})\pi_0$;

DR.9 $\overline{D[f]} = 1 \times \overline{f}$ **(Undefinedness comes from the "point")**.

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Example 1: Smooth Maps on open subsets of $\mathbb{R}^n$

- The Jacobian matrix provides the differential structure.

$$J_f(y_1,\dots,y_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(y_1,\dots,y_n) & \cdots & \frac{\partial f_1}{\partial x_n}(y_1,\dots,y_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(y_1,\dots,y_n) & \cdots & \frac{\partial f_m}{\partial x_n}(y_1,\dots,y_n) \end{bmatrix}$$

- $D[f] : (x_1,\dots,x_n,y_1,\dots,y_n) \mapsto J_f(y_1,\dots,y_n)\cdot(x_1,\dots,x_n)$.

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Example 2: Rational Functions Let R be a rig. Then $\textsc{Rat}_R$ is a restriction category where

Obj: $n \in \mathbb{N}$

Arr: $n \to m$ given by a pair $\left(\left(\frac{f_i}{g_i}\right)_{i=1}^{m}, \mathcal{U}\right)$ where $\mathcal{U}$ is a finitely generated multiplicative set with $\frac{f_i}{g_i} \in R[x_1, \ldots, x_n]\left[\mathcal{U}^{-1}\right]$.

Id: $((x_i), \emptyset) : n \longrightarrow n$

Comp: By substitution

Rest: $\overline{\left(\left(\frac{p_i}{q_i}\right)_{i=1}^{m}, \mathcal{U}\right)} = ((x_i), \mathcal{U})$

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For polynomials over a rig, there is a formal partial derivative. Let $f = \sum_l a_l x_1^{l_1} \cdots x_n^{l_n}$. Then the partial derivative with respect to x_k is,

$$\frac{\partial f}{\partial x_k} = \sum_l l_k a_l x_1^{l_1} \cdots x_{k-1}^{l_{k-1}} x_k^{l_k - 1} x_{k+1}^{l_{k+1}} \cdots x_n^{l_n}$$

If R is a ring, then rational functions also have a formal partial derivative.

$$\frac{\partial \frac{p}{q}}{\partial x_k} = \frac{\frac{\partial p}{x_k} q - p \frac{\partial q}{x_k}}{q^2}.$$

The differential on RAT_R is given by the formal Jacobian matrix of these formal partial derivatives.

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Join Restriction Categories

- We have just seen two examples of differential restriction categories. There is a difference: one has topological properties the other does not. We will explore the structure that gives these topological properties.
- We will also give a differential restriction category that is not defined by a Jacobian matrix.

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Definition 7. *In a restriction category, parallel maps f and g are* **compatible** *if $\overline{f}\, g = \overline{g}\, f$.*

Definition 8. *A restriction category, $\mathbb{X}$, is a* **join restriction category** *if every set of compatible maps, $C \subseteq \mathbb{X}(A, B)$, has a join (sup) that is stable; i.e.,*

$$f\left(\bigvee_{g\in C} g\right) = \bigvee_{g\in C} fg.$$

Theorem 1. *Join and differential restriction structure are compatible; i.e.*

$$D\left[\bigvee_i f_i\right] = \bigvee D\left[f_i\right].$$

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Smooth functions are a differential join restriction category, but rational functions are not. Rational functions can have a sup for every set of compatible maps, but stability fails. Consider,

$$(1, \langle x-1 \rangle) \smile (1, \langle y-1 \rangle),$$

so the join must be

$$(1, \langle 1 \rangle).$$

As a counterexample, consider the substitution $\left[x^2/x, x^2/y\right]$. $\langle x-1 \rangle \cap \langle y-1 \rangle$ does not contain x or y; thus, the substitution does not contain $x-1$. However,

$$x - 1 \in \left(\left[x^2/x, x^2/y\right] \langle x-1 \rangle \cap \left[x^2/x, x^2/y\right] \langle y-1 \rangle\right).$$

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- The structure of join restriction categories allow any map to be broken into arbitrary pieces and put together again.
- Join restriction categories have more topological structure; for an object A, $\{e : A \to A \mid e = \overline{e}\}$ is a locale.
- Join restriction categories allow the classical completion [3] and the manifold completion [4].

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Join Completion

Let $\mathbb{X}$ be any restriction category. We can obtain a join restriction category from $\mathbb{X}$ by a universal construction $Jn(\mathbb{X})$:

Obj: Those of $\mathbb{X}$.

Arr: $A \xrightarrow{\mathcal{F}} B$ is a subset $\mathcal{F} \subseteq \mathbb{X}(A, B)$ that is pairwise compatible and has the property that if $f \in \mathcal{F}$ and $h \leq f$ (i.e. $\overline{h}\, f = h$) then $h \in \mathcal{F}$.

Id: $\downarrow 1_A = \{d : A \to A \mid d \leq 1_A\}$

Comp: $\mathcal{F}\mathcal{G} = \{fg \mid f \in \mathcal{F}, g \in \mathcal{G}\}$

Rest: $\overline{\mathcal{F}} = \{\overline{f} \mid f \in \mathcal{F}\}$

Join: $\bigvee_i \mathcal{F}_i = \bigcup_i \mathcal{F}_i$

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Theorem 2. *If $\mathbb{X}$ is a differential restriction category, then $Jn(\mathbb{X})$ is a differential join restriction category.*

The differential structure on $Jn(\mathbb{X})$ is

$$\begin{aligned} D[\mathcal{F}] &= \downarrow \{D[f] \mid f \in \mathcal{F}\} \\ &= \{e \mid e \leq D[f] \text{ for some } f \in \mathcal{F}\} \end{aligned}$$

- This differential restriction structure is not given by a Jacobian.
- We can now obtain a differential join restriction category from RAT_R.

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Concluding Remarks

- From any differential restriction category we can obtain a differential join restriction category.
- Coming next: The manifold completion of a differential join restriction category has a tangent bundle structure which allow the axiomatization of differential geometry categories.

Thank you.

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References

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