Groupoidification and the Hecke Bicategory: A framework for geometric representation theory

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- Geometric Representation Theory
- Degroupoidification
- The Hecke Bicategory
- Example: The A_2 Hecke algebra

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Outline

Geometric Representation Theory

Degroupoidification

Bicategories of Spans

Example: The A_2 Hecke algebra

A great deal of representation theory can be realized geometrically via convolution products on various homology theories.

The basic idea is that finite-dimensional irreducible representations of certain Coxeter groups and Lie and associative algebras can be obtained by "pull-tensor-push" operations or "integral transforms".

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Toy Example

Given a span of finite sets

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

and a function $K \in \mathbb{C}^S$, we can construct a linear operator, or integral transform,

$$K * - \colon \mathbb{C}^X \to \mathbb{C}^Y$$

defined as

$$q_*(K \cdot p^*(f))(y) = \sum_{s \in q^{-1}(y)} K(s) \cdot f(p(s)).$$

Orlov's Result

In our toy example we have the isomorphism

$$\mathbb{C}^{(X\times Y)} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C}^Y)$$

For Fourier-Mukai transforms, the derived version of a correspondence, we have Orlov's result, which roughly states that for smooth projective varieties

$$D^b(X\times Y) \simeq \mathrm{Hom}(D^b(X), D^b(Y))$$

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Some Geometric Theories

Our toy example illustrates the "pull-tensor-push" philosophy of integral transforms.

More sophisticated examples:

- Convolution algebras on
  - Borel-Moore homology
  - equivariant K-theory
  - constructible functions
- Correspondences in the product of Hilbert schemes
- Fourier-Mukai transforms between derived categories
- The theory of motives

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Categorification and Matrix Multiplication

There is momentum in geometric representation theory towards *geometric function theory*, which might be considered the study of *higher* geometric representation theory.

Geometric function theory considers notions of higher generalized functions on higher generalized spaces such as groupoids, orbifolds and stacks, such that all of the *generalized linear maps* between the functions on two spaces arise from a higher analog of plain matrix multiplication, namely from a pull-tensor-push operation. (Loosely quoted from the nLab.)

Categorification

It is useful to provide a unified framework in which to formalize and compare these geometric function theories. To this end, we want to consider the pull-tensor-push operations along with appropriate homology theories as decategorification functors.

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Groupoidification is a categorification theory designed to study geometric constructions in representation theory.

vector spaces $\rightsquigarrow$ groupoids

linear operators $\rightsquigarrow$ spans of groupoids

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Degroupoidification

The **degroupoidification** functor

$$\mathcal{D}\colon \mathrm{Span}(\mathrm{Grpd}) \to \mathrm{Vect}$$

takes a groupoid X to the vector space $\mathcal{D}(X) := \mathbb{C}^{\underline{X}}$, where $\underline{X}$ is the set of isomorphism classes of X, and a span of groupoids

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

to a linear operator

$$\mathcal{D}(S)\colon \mathbb{C}^{\underline{X}} \to \mathbb{C}^{\underline{Y}}.$$

Key to Decategorification

Each geometric theory has key technical results or tools from which we obtain the relevant algebraic structure constants. For example, geometric constructions of irreducible representations of $\mathcal{U}(\mathfrak{sl}(n))$ arise, in part, from the Euler characteristic of flag varieties.

Groupoid Cardinality

$$|X| = \sum_{[x]\in \underline{X}} \frac{1}{|Aut(x)|}$$

Example

Let E be the groupoid of finite sets.

$$|E| = \sum_{[e]\in E} \frac{1}{|Aut(e)|} = \sum_{n\in\mathbb{N}} \frac{1}{|S_n|} = \sum_{n\in\mathbb{N}} \frac{1}{n!} = e.$$

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Linear Operators from Spans

Given a span S and a basis element $[x] \in \underline{X}$

$$\begin{tikzcd}
 & & S1 \arrow[dl, "\pi_S"'] \arrow[dr] & \\
 & S \arrow[dl, "q"'] \arrow[dr, "p"] & & 1 \arrow[dl, "x"] \\
Y & & X &
\end{tikzcd}$$

we define

$$\mathcal{D}(S)(x) = \sum_{[y] \in \underline{Y}} |(q\pi_S)^{-1}(y)|[y] \in \mathbb{C}^{\underline{Y}}.$$

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Categorified Linear Algebra

To find a good framework for categorified representation theory, it makes sense to, as the King is so often quoted,

> "Begin at the beginning, and go on till you come to the end: then stop"

The first tool of representation theory is, of course, linear algebra. So we would like to develop solid foundations of categorified linear algebra.

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Concrete and Abstract Vector Spaces

Groupoidification models the theory of concrete vector spaces with spans of groupoids replacing linear maps.

A closely related monoidal 2-category is the underlying 2-category of topos frames. Here we forget the structure of a bounded topos and consider the cocontinuous functors between cocomplete categories (everything over Set).

Some help from the audience?

This is not quite the right setting for categorified abstract linear algebra.

Nonetheless, this is the right type of setting to find a version Orlov's result.

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Plodding ahead...

Further, we can categorify permutation representations of a finite group in this setting of spans of groupoids and cocontinuous functors between presheaf topoi.

The categorification of permutation representations is an enriched bicategory which as a corollary categorifies the Hecke algebra.

We study this to get some intuition for building a nice framework for geometric representation theory.

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Permutation Representations

Given a finite group G, the category of permutation representations $\mathrm{PermRep}(G)$ consists of

- finite-dimensional representations of G with a chosen basis fixed by the action of G, and
- G-equivariant linear operators.

This is a Vect-enriched category.

So we want to work as much as possible at the enriched level of categorified linear algebra.

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Bicategories of Spans

Theorem

Given a bicategory $\mathcal{B}$ with pullbacks, finite limits and all 2-cells invertible, there is a monoidal bicategory $\mathrm{Span}(\mathcal{B})$.

$\mathrm{Span}(\mathrm{Grpd})$ is a monoidal bicategory.

Change of Base Functors

Categorified Linearization

There is a monoidal functor $\mathrm{Span}(\mathrm{Grpd})$ to Cocont defined by

$$X \mapsto \mathrm{Set}^X$$

$$(Y \xleftarrow{q} S \xrightarrow{p} X) \mapsto q_! p^* \colon \mathrm{Set}^X \to \mathrm{Set}^Y$$

(and taking maps of spans to natural transformations.)

Degroupoidification

Degroupoidification is a monoidal functor from $\mathrm{Span}(\mathrm{Grpd})$ to Vect.

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Enriched Bicategories

Since we are enriching over groupoids and spans of groupoids, we need a concept of enriched bicategories.

Enriched Bicategories

Given a monoidal bicategory $\mathcal{V}$, a $\mathcal{V}$-enriched bicategory consists of

- a set of objects x,y,z,. . . ,
- for each pair x,y, a $\mathcal{V}$-object of morphisms $\hom(x, y)$,
- for each triple of objects x,y,z, a $\mathcal{V}$-morphism called *composition*

$$\hom(x, y) \otimes \hom(y, z) \to \hom(x, z),$$

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- for each quadruple w,x,y,z, an invertible $\mathcal{V}$-2-morphism called the *associator*

$$\begin{tikzcd}
((w,x)\otimes(x,y))\otimes(y,z) \arrow[rr, "a"'] \arrow[d, "c\otimes 1"'] & & (w,x)\otimes((x,y)\otimes(y,z)) \arrow[d, "1\otimes c"] \\
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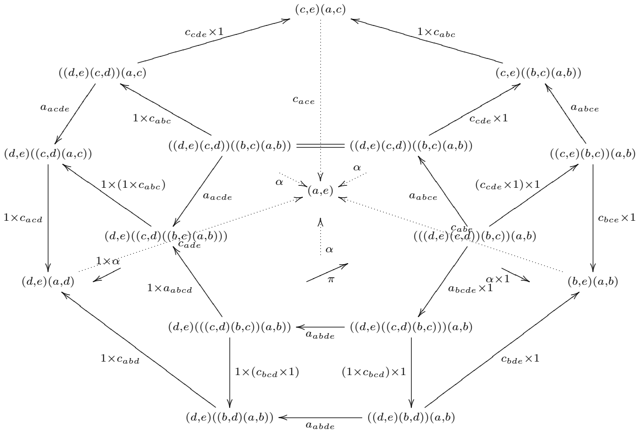
One of the Axioms



Change of Base

Change of base will provide a means of lifting our decategorification functor to the enriched setting as well as switching between the span of groupoids and cocontinuous functor pictures.

Change of Base

Given a $\mathcal{V}$-enriched bicategory $\mathcal{B}$ and a lax monoidal functor $f\colon \mathcal{V} \to \mathcal{W}$, then there is a $\mathcal{W}$-enriched bicategory $\bar{f}(\mathcal{B})$.

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The Hecke Bicategory

For each finite group G, there is a $\mathrm{Span}(\mathrm{Grpd})$-enriched bicategory $\mathrm{Hecke}(G)$ consisting of

- finite G-sets (think permutation representation) as objects,
- for each X, Y, a hom-groupoid

$$(X \times Y)//G$$

  called the *action groupoid*,

- for each triple X, Y, Z, a composition span

$$\begin{tikzcd} & (X \times Y \times Z) \arrow[dl, "\pi_{13}"'] \arrow[dr, "\pi_{12} \times \pi_{23}"] & \\ (X \times Z) & & (X \times Y) \times (Y \times Z) \end{tikzcd}$$

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A Categorification Theorem

For each finite group G, there is an equivalence of Vect-enriched categories

$$\bar{\mathcal{D}}(\mathrm{Hecke}(G)) \simeq \mathrm{PermRep}(G)$$

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The Cocont-enriched bicategory

Passing from $\mathrm{Hecke}(G)$ to the Cocont-enriched bicategory by change of base, we obtain a more hands-on description which is more or less $\mathrm{Span}(\mathrm{GSet})$ consisting of

- finite G-sets $X, Y, Z, \ldots$,
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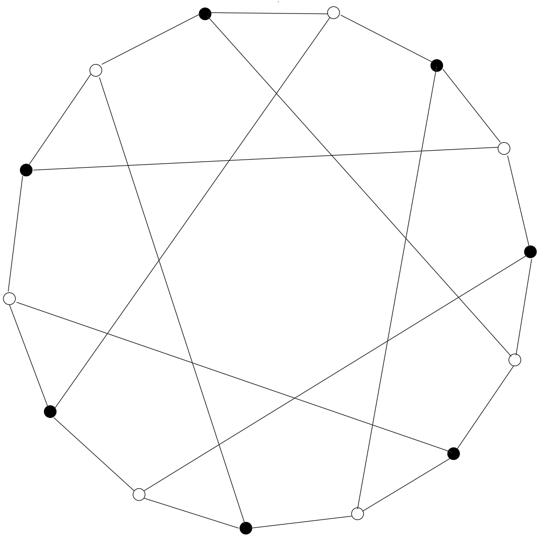
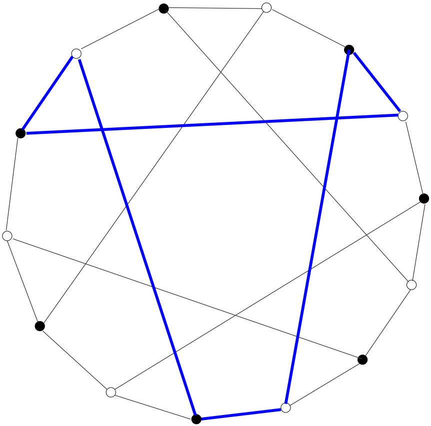
The A_2 Building over the Field of 2 Elements



The S_3 Apartment

The Hexagon

S_3 is the Weyl group of $G = SL(3, \mathbb{F}_2)$ and this building is the G-set of flags $X = G/B$, where B is the Borel subgroup of upper triangular matrices.



Special Spans

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

$$P = \{((p,l),(p',l) \in X \times X \mid p \neq p'\}$$

$$L = \{((p,l),(p,l') \in X \times X \mid l \neq l'\}$$

The spans P and L satisfy the relations of the A_2 Hecke algebra up to isomorphism.

The Categorified Hecke Algebra

The Hecke Algebra

The Hecke algebra the associative algebra with generated by P and L with relations:

$$PLP = LPL$$

given by the existence of hexagonal apartments and

$$P^2 = (q-1)P + q, \quad L^2 = (q-1)L + q$$

which comes from counting points in projective geometry.