# Π- and Σ-types in homotopy theoretic models of type theory

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1. Type theory
2. Lccc semantics
3. Homotopy-theoretic semantics
4. Examples

## What are Π- and Σ-types?

| Type theory | Logic | Set theory |
|---|---|---|
| $\Pi_{x:\,A} B(x)$ | $\forall_{x \in A}\ \varphi$ | $\prod_{x \in X} Y_x$ |
| $\Sigma_{x:\,A} B(x)$ | $\exists_{x \in A}\ \varphi$ | $\coprod_{x \in X} Y_x$ |

## Π-types

$$\frac{\Gamma,\ x\colon A \vdash B(x)\ \text{type}}{\Gamma \vdash \Pi_{x\colon A} B(x)\ \text{type}}\ \Pi\text{-}\textsc{form}$$

$$\frac{\Gamma,\ x\colon A \vdash B(x)\ \text{type} \qquad \Gamma,\ x\colon A \vdash b(x)\colon B(x)}{\Gamma \vdash \lambda x.b(x)\colon \Pi_{x\colon A} B(x)}\ \Pi\text{-}\textsc{intro}$$

$$\frac{\Gamma \vdash f\colon \Pi_{x\colon A} B(x) \qquad \Gamma \vdash a\colon A}{\Gamma \vdash \text{app}(f,\ a)\colon B(a)}\ \Pi\text{-}\textsc{elim}$$

$$\frac{\begin{array}{c}\Gamma,\ x\colon A \vdash B(x)\ \text{type} \\ \Gamma,\ x\colon A \vdash b(x)\colon B(x) \qquad \Gamma \vdash a\colon A\end{array}}{\Gamma \vdash \text{app}(\lambda x.b(x),\ a) = b(a)\colon B(a)}\ \Pi\text{-}\textsc{comp}$$

# Σ-types

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma,\ x\colon A \vdash B(x) \text{ type}}{\Gamma \vdash \Sigma_{x\colon A} B(x) \text{ type}}\ \Sigma\text{-FORM}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma,\ x\colon A \vdash B(x) \text{ type}}{\Gamma,\ x\colon A,\ y\colon B(x) \vdash \mathsf{pair}(x,\ y)\colon \Sigma_{x\colon A} B(x)}\ \Sigma\text{-INTRO}$$

$$\frac{\begin{array}{c}\Gamma,\ z\colon \Sigma_{x\colon A} B(x) \vdash C(z) \text{ type} \\ \Gamma,\ x\colon A,\ y\colon B(x) \vdash d(x,y)\colon C(\mathsf{pair}(x,\ y))\end{array}}{\Gamma,\ z\colon \Sigma_{x\colon A} B(x) \vdash \mathsf{split}_d(z)\colon C(z)}\ \Sigma\text{-ELIM}$$

$$\frac{\begin{array}{c}\Gamma,\ z\colon \Sigma_{x\colon A} B(x) \vdash C(z) \text{ type} \\ \Gamma,\ x\colon A,\ y\colon B(x) \vdash d(x,y)\colon C(\mathsf{pair}(x,\ y))\end{array}}{\Gamma,\ x\colon A,\ y\colon B(x) \vdash \mathsf{split}_d(z) = d(x,y)\colon C(\mathsf{pair}(x,\ y))}\ \Sigma\text{-COMP}$$

## Example

For

$$n\colon \mathsf{Nat} \vdash \mathbb{R}^n \text{ type}$$

we can form:

$$\Sigma_{n\colon \mathsf{Nat}}\mathbb{R}^n$$

and

$$\Pi_{n\colon \mathsf{Nat}}\mathbb{R}^n.$$

## Some notation

Let $\mathbb{C}$ be a category with pullbacks and $f\colon B \longrightarrow A$ a morphism in $\mathbb{C}$. Then

$$\Delta_f\colon \mathbb{C}/A \longrightarrow \mathbb{C}/B$$

will denote the pullback functor along $f$.
This functor has a left adjoint

$$\Sigma_f\colon \mathbb{C}/B \longrightarrow \mathbb{C}/A$$

mapping $x\colon X \longrightarrow B$ to its composition with $f$ that is to the object $f \circ x\colon X \longrightarrow B \longrightarrow A$ in $\mathbb{C}/A$.

## Locally cartesian closed categories

**Fact**

A category $\mathbb{C}$ is locally cartesian closed if and only if it has all finite limits and for each $f : B \longrightarrow A$ the pullback functor $\Delta_f$ has a right adjoint.

The right adjoint to $\Delta_f$ (if it exists) will be denoted by $\Pi_f$. So for any morphism $f : B \longrightarrow A$ in a locally cartesian closed category $\mathbb{C}$ we have three functors associated to it:

$$\Sigma_f \dashv \Delta_f \dashv \Pi_f .$$

## Interpretation

Semantics for Π-types:

$$\begin{array}{ccc} \Gamma.A.B & & \Gamma.\Pi_{x:\,A}B \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\Pi_\alpha\beta} \\ \Gamma.A & \xrightarrow[\alpha]{} & \Gamma \end{array}$$

Semantics for Σ-types:

$$\begin{array}{ccc} \Gamma.A.B & & \Gamma.\Sigma_{x:\,A}B \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\Sigma_\alpha\beta=\alpha\circ\beta} \\ \Gamma.A & \xrightarrow[\alpha]{} & \Gamma \end{array}$$

**Disappointment**

This semantics is extensional!

By interpreting types as fibrations, we get an intensional semantics (for $\mathrm{Id}$-types). However there is a:

**Question**

How to fit Π- and Σ-types into this interpretation?

Σ is as easy as pie (since it can be interpreted as composition) but Π is more complicated.

There is:

**Good news!**

We need $\Pi_f$ to exist only for $f$ being a fibration.

and also:

**Bad news. . .**

$\Pi_f$ has to preserve fibrations.

What can we do with that?

Observe that if $\mathbb{C}$ is a model category, then so is any slice of $\mathbb{C}$ with the induced model structure.
Now we may apply the following theorem:

**Theorem**

*Let $\mathbb{C}$ be a model category and $f : B \longrightarrow A$ a morphism in $\mathbb{C}$. Then $\Pi_f$ preserves fibrations if and only if $\Delta_f$ preserves cofibrations and trivial cofibrations.*

**Corollary**

*If $\mathbb{C}$ is right proper (i.e. weak equivalences are stable under pullback), cofibrations in $\mathbb{C}$ are stable under pullback, and $\Pi_f$ exists for any fibration $f$ in $\mathbb{C}$, then $\mathbb{C}$ has models of $\Pi$-types.*

## Groupoids

The category **Gpd** of groupoids has a structure of a model category with:

- fibrations = Grothendieck fibrations,
- cofibrations = functors injective on objects,
- weak equivalences = categorical equivalences.

## Simplicial sets

The category **SSets** of simplicial sets has a structure of a model category with:

- fibrations = Kan fibrations,
- cofibrations = monomorphisms,
- weak equivalences = maps such that the induced map of geometric realizations is a homotopy equivalence of topological spaces.

BTW. The category of simplicial presheaves is also a model of Martin-Löf Type Theory.

## Preorders

The category **PreOrd** of preorders has a structure of a model category (the restriction of Joyal's model structure on **Cat**) with:

- fibrations = isofibrations,
- cofibrations = functors injective on objects,
- weak equivalences = categorical equivalences.

## Future research

- further examples (eg. from algebraic geometry),
- study of homotopy limits and colimits by means of type theory.

# Thank you!

(You can wake up)