Finite dimensional Hilbert spaces are complete for dagger compact closed categories (extended abstract)

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Abstract

We show that an equation follows from the axioms of dagger compact closed categories if and only if it holds in finite dimensional Hilbert spaces.

Keywords: Dagger compact closed categories, Hilbert spaces, completeness.

1 Introduction

Hasegawa, Hofmann, and Plotkin recently showed that the category of finite dimensional vector spaces over any fixed field k of characteristic 0 is *complete* for traced symmetric monoidal categories [2]. What this means is that an equation holds in all traced symmetric monoidal categories if and only if it holds in finite dimensional vector spaces. Via Joyal, Street, and Verity's "Int"-construction [3], it is a direct corollary that finite dimensional vector spaces are also complete for compact closed categories. The present paper makes two contributions: (1) we simplify the proof of Hasegawa, Hofmann, and Plotkin's result, and (2) we extend it to show that finite dimensional *Hilbert* spaces are complete for *dagger* traced symmetric monoidal categories (and hence for dagger compact closed categories).

2 Statement of the main result

For a definition of dagger compact closed categories, their term language, and their graphical language, see [1,4]. We also use the concept of a *dagger traced monoidal category*, which is a dagger symmetric monoidal category [4] with a trace operation

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[3] satisfying $\mathrm{Tr}^X_{U,V}(f)^\dagger = \mathrm{Tr}^X_{V,U}(f^\dagger)$. We note that every dagger compact closed category is also dagger traced monoidal; conversely, by Joyal, Street, and Verity's "Int" construction, every dagger traced monoidal category can be fully embedded in a dagger compact closed category.

We will make use of the soundness and completeness of the graphical representation, specifically of the following result:

Theorem 2.1 ([4]) *A well-typed equation between morphisms in the language of dagger compact closed categories follows from the axioms of dagger compact closed categories if and only if it holds, up to graph isomorphism, in the graphical language.*

An analogous result also holds for dagger traced monoidal categories.

The goal of this paper is to prove the following:

Theorem 2.2 *Let $M, N : A \to B$ be two terms in the language of dagger compact closed categories. Suppose that $[\![M]\!] = [\![N]\!]$ for every possible interpretation (of object variables as spaces and morphism variables as linear maps) in finite dimensional Hilbert spaces. Then $M = N$ holds in the graphical language (and therefore, holds in all dagger compact closed categories).*

3 Reductions

Before attempting to prove Theorem 2.2, we reduce the statement to something simpler. By arguments analogous to those of Hasegawa, Hofmann, and Plotkin [2], it suffices without loss of generality to consider terms M, N that satisfy some additional conditions. The additional conditions are:

- We may assume that $M, N : I \to I$, i.e., that both the domain and codomain of M and N are the tensor unit. Such terms are called *closed*. The restriction to closed terms is without loss of generality, because given general $M, N : A \to B$, we can extend the language with two new morphism variables $f : I \to A$ and $g : B \to I$, and apply the theorem to the terms $M' = g \circ M \circ f$ and $N' = g \circ N \circ f$. Since g, f are new symbols, $g \circ M \circ f = g \circ N \circ f$ in the graphical language implies that $M = N$ in the graphical language.
- It suffices to consider terms M, N in the language of *dagger traced monoidal* categories. Namely, by Joyal, Street, and Verity's "Int"-construction [3], every statement about dagger compact closed categories can be translated to an equivalent statement about dagger traced monoidal categories. This is done by eliminating occurrences of the *-operation: one replaces every morphism variable such as $f : A^* \otimes B \otimes C^* \to D^* \otimes E$ by an equivalent new morphism variable such as $f' : B \otimes D \to A \otimes C \otimes E$ that does not use the *-operation.
- It suffices to consider terms whose graphical representation does not contain any "trivial cycles". Trivial cycles are connected components of a diagram that do not contain any morphism variables, such as the ones obtained from the trace of an identity morphism. The restriction is without loss of generality because if M, N have different numbers or types of trivial cycles, they can be easily separated in Hilbert spaces [2]. We say that a diagram is *simple* if it contains no trivial cycles.

4 Outline of the result by examples

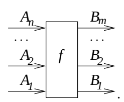
The formal statement and proof of Theorem 2.2 requires a fair amount of notation. Nevertheless, the main idea is simple, and is perhaps better illustrated in an example. We thus start by giving an informal explanation of the proof in this section, based on examples. The full technical proof is given in Section 5.

4.1 Signatures, diagrams, and interpretations

We assume given a set of *object variables*, denoted A, B etc., and a set of *morphism variables*, denoted f, g etc. A *sort* $\boldsymbol{A}$ is a finite sequence of object variables. We usually write $A_1 \otimes \ldots \otimes A_n$ for an n-element sequence, and I for the empty sequence. We assume that each morphism variable f is assigned two fixed sorts, called its *domain* $\boldsymbol{A}$ and *codomain* $\boldsymbol{B}$ respectively, and we write $f : \boldsymbol{A} \to \boldsymbol{B}$. We further require a fixpoint-free involution $(-)^\dagger$ on the set of morphism variables, such that $f^\dagger : \boldsymbol{B} \to \boldsymbol{A}$ when $f : \boldsymbol{A} \to \boldsymbol{B}$.

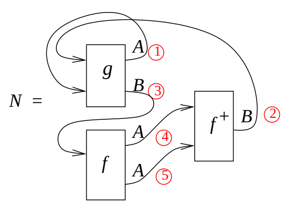
The collection of object variables and morphism variables, together with the domain and codomain information and the dagger operation is called a *signature* Σ of dagger monoidal categories.

Graphically, we represent a morphism variable $f : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_m$ as a box



The wires on the left are called the *inputs* of f, and the wires on the right are called its *outputs*. Note that each box is labeled by a morphism variable, and each wire is labeled by an object variable.

A (closed simple dagger symmetric traced monoidal) *diagram* over a signature Σ consists of zero or more boxes of the above type, all of whose wires have been connected in pairs, such that each connection is between the output wire of some box and the input wire of some (possibly the same, possible another) box. Here is an example of a diagram N over the signature given by $f : B \to A \otimes A$, $g : A \otimes B \to B \otimes A$.



In the illustration, we have numbered the wires 1 to 5 to aid the exposition below; note that this numbering is not formally part of the diagram.

An *interpretation* of a signature in finite-dimensional Hilbert spaces consists of the following data: for each object variable A, a chosen finite-dimensional Hilbert space $[\![A]\!]$, and for each morphism variable $f : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_m$, a chosen linear map $[\![f]\!] : [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!] \to [\![B_1]\!] \otimes \ldots \otimes [\![B_m]\!]$, such that $[\![f^\dagger]\!] = [\![f]\!]^\dagger$.

The *denotation* of a diagram M under a given interpretation is a scalar that is defined by the usual "summation over internal indices" formula. For example, the denotation of the above diagram N is:

$$[\![N]\!] = \sum_{a_1, b_2, b_3, a_4, a_5} [\![g]\!]^{a_1, b_2}_{b_3, a_1} \cdot [\![f]\!]^{b_3}_{a_5, a_4} \cdot [\![f^\dagger]\!]^{a_5, a_4}_{b_2}. \qquad (4.1)$$

Here a_1, a_4, a_5 range over some orthonormal basis of $[\![A]\!]$, b_2, b_3 range over some orthonormal basis of $[\![B]\!]$, and $[\![f]\!]^{b_3}_{a_5, a_4}$ stands for the matrix entry $\langle a_5 \otimes a_4 \mid [\![f]\!](b_3) \rangle$. As is well-known, this denotation is independent of the choice of orthonormal bases.

4.2 Proof sketch

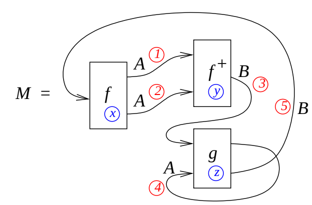
By the reductions in Section 3, Theorem 2.2 is a consequence of the following lemma:

Lemma 4.1 (Relative completeness) *Let M be a (closed simple dagger traced monoidal) diagram. Then there exists an interpretation $[\![-]\!]_M$ in finite dimensional Hilbert spaces, depending only on M, such that for all N, $[\![N]\!]_M = [\![M]\!]_M$ holds if and only if N and M are isomorphic diagrams.*

Clearly, the right-to-left implication is trivial, for if N and M are isomorphic diagrams, then $[\![N]\!] = [\![M]\!]$ holds under every interpretation; their corresponding summation formulas differ at most by a reordering of summands and factors. It is therefore the left-to-right implication that must be proved.

The general proof of this lemma requires a fair amount of notation, as well as more careful definitions than we have given above. A full proof appears in Section 5 below. Here, we illustrate the proof technique by means of an example.

Take the same signature as above, and suppose M is the following diagram:



Again, we have numbered the wires from 1 to 5, and this time, we have also numbered the boxes x, y, and z.

We must now construct the interpretation required by the Lemma. It is given as follows. Define $[\![A]\!]_M$ to be a 3-dimensional Hilbert space with orthonormal basis $\{A_1, A_2, A_4\}$. Define $[\![B]\!]_M$ to be a 2-dimensional Hilbert space with orthonormal basis $\{B_3, B_5\}$. Note that the names of the basis vectors have been chosen to suggest a correspondence between basis vectors of $[\![A]\!]_M$ and wires labeled A in the diagram M, and similarly for $[\![B]\!]_M$.

Let x, y, and z be three algebraically independent transcendental complex numbers. This means that x, y, z do not satisfy any polynomial equation $p(x, y, z, \bar{x}, \bar{y}, \bar{z}) = 0$ with rational coefficients, unless $p \equiv 0$.

Define three linear maps $F_x : [\![B]\!]_M \to [\![A]\!]_M \otimes [\![A]\!]_M$, $F_y : [\![A]\!]_M \otimes [\![A]\!]_M \to [\![B]\!]_M$, and $F_z : [\![A]\!]_M \otimes [\![B]\!]_M \to [\![B]\!]_M \otimes [\![A]\!]_M$ as follows. We give each map by its matrix representation in the chosen basis.

$$(F_x)^i_{jk} = \begin{cases} x & \text{if } i = B_5,\ j = A_2, \text{ and } k = A_1, \\ 0 & \text{else,} \end{cases}$$
$$(F_y)^{ij}_k = \begin{cases} y & \text{if } i = A_2,\ j = A_1, \text{ and } k = B_3, \\ 0 & \text{else,} \end{cases}$$
$$(F_z)^{ij}_{kl} = \begin{cases} z & \text{if } i = A_4,\ j = B_3,\ k = B_5, \text{ and } l = A_4, \\ 0 & \text{else.} \end{cases}$$

It is hopefully obvious how each of these linear functions is derived from the diagram M: each matrix contains precisely one non-zero entry, whose position is determined by the numbering of the input and output wires of the corresponding box in M.

The interpretations of f and g are then defined as follows:

$$[\![f]\!]_M = F_x + F_y^\dagger, \qquad [\![g]\!]_M = F_z.$$

Note that we have taken the adjoint of the matrix F_y, due to the fact that the corresponding box was labeled $f^\dagger$. This finishes the definition of the interpretation $[\![-]\!]_M$. It can be done analogously for any diagram M.

To prove the condition of the Lemma, we first observe that the interpretation $[\![N]\!]_M$ of any diagram N is given by a summation formula analogous to (4.1). Moreover, from the definition of the interpretation $[\![-]\!]_M$, it immediately follows that the scalar $[\![N]\!]_M$ can be (uniquely) expressed as a polynomial $p(x, y, z, \bar{x}, \bar{y}, \bar{z})$ with integer coefficients in the variables x, y, z and their complex conjugates. We also note that this polynomial is homogeneous, and its degree is equal to the number of boxes in N.

We claim that the coefficient of p at xyz is non-zero if and only if N is isomorphic to M. The proof is a direct calculation, using (4.1) and the definition of $[\![-]\!]_M$. Essentially, any non-zero contribution to xyz in the summation formula must come from a choice of a basis vector $A_{\psi(w)}$ of $[\![A]\!]_M$ for each wire w labeled A in N, and a choice of a basis vector $B_{\psi(w)}$ of $[\![B]\!]_M$ for each wire w labeled B in N, together with a bijection ϕ between the boxes of N and the set $\{x, y, z\}$; moreover, the contribution can only be non-zero if the choice of basis vectors is "compatible" with the bijection ϕ. Compatibility amounts precisely to the requirement that the maps ψ and ϕ determine a graph isomorphism from N to M. For example, in the calculation of $[\![N]\!]_M$ according to equation (4.1), the only non-zero contribution to xyz in p comes from the assignment $a_1 \mapsto A_4$, $b_2 \mapsto B_3$, $b_3 \mapsto B_5$, $a_4 \mapsto A_1$, and $a_5 \mapsto A_2$, which corresponds exactly to the (in this case unique) isomorphism from N to M.

In fact, we get a stronger result: the integer coefficient of p at xyz is equal to the *number* of different isomorphisms between N and M (usually 0 or 1, but it could be higher if M has non-trivial automorphisms).

5 Technical development

5.1 Signatures and diagrams

Definition 5.1 (Signature) A *signature* of dagger monoidal categories is a quintuple $\Sigma = \langle \mathrm{Obj}, \mathrm{Mor}, \mathrm{dom}, \mathrm{cod}, \dagger \rangle$ consisting of:

- a set Obj of *object variables*, denoted $A, B, C, \ldots$;
- a set Mor of *morphism variables*, denoted $f, g, h, \ldots$;
- functions $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor} \to \mathrm{Obj}^*$, called the *domain* and *codomain* functions, respectively, where Obj^* is the set of finite sequences of object variables;
- an operation $\dagger : \mathrm{Mor} \to \mathrm{Mor}$, such that for all $f \in \mathrm{Mor}$, $f^{\dagger\dagger} = f$, $f^\dagger \neq f$, $\mathrm{dom}\, f^\dagger = \mathrm{cod}\, f$, and $\mathrm{cod}\, f^\dagger = \mathrm{dom}\, f$.

As before, we write a sequence of n object variables as $A_1 \otimes \ldots \otimes A_n$, or as $\boldsymbol{A}$, and we write $|\boldsymbol{A}| = n$ for the length of a sequence. We write $f : \boldsymbol{A} \to \boldsymbol{B}$ if $\mathrm{dom}\, f = \boldsymbol{A}$ and $\mathrm{cod}\, f = \boldsymbol{B}$.

Definition 5.2 (Diagram) A *(closed simple dagger symmetric traced monoidal) diagram* $M = \langle W^M, B^M, \ell^M_{\mathrm{w}}, \ell^M_{\mathrm{b}}, \theta^M_{\mathrm{in}}, \theta^M_{\mathrm{out}} \rangle$ over a signature Σ consists of the following:

- a set W^M of *wires*;
- a set B^M of *boxes*;
- a pair of labeling functions $\ell^M_{\mathrm{w}} : W^M \to \mathrm{Obj}$ and $\ell^M_{\mathrm{b}} : B^M \to \mathrm{Mor}$;
- a pair of bijections $\theta^M_{\mathrm{in}} : \mathrm{Inputs}^M \to W^M$ and $\theta^M_{\mathrm{out}} : \mathrm{Outputs}^M \to W^M$, where

$$\begin{aligned} \mathrm{Inputs}^M &= \{(i, b) \mid b \in B^M, n = |\mathrm{dom}(\ell^M_{\mathrm{b}}(b))|, 1 \leqslant i \leqslant n\}, \\ \mathrm{Outputs}^M &= \{(b, j) \mid b \in B^M, m = |\mathrm{cod}(\ell^M_{\mathrm{b}}(b))|, 1 \leqslant j \leqslant m\}. \end{aligned}$$

Moreover, a diagram is required to satisfy the following typing conditions:

- whenever $b \in B^M$, $f = \ell^M_{\mathrm{b}}(b)$, $\boldsymbol{A} = \mathrm{dom}\, f$, $(i, b) \in \mathrm{Inputs}^M$, $\theta^M_{\mathrm{in}}(i, b) = w$, then $\ell^M_{\mathrm{w}}(w) = A_i$, and
- whenever $b \in B^M$, $f = \ell^M_{\mathrm{b}}(b)$, $\boldsymbol{B} = \mathrm{cod}\, f$, $(b, j) \in \mathrm{Outputs}^M$, $\theta^M_{\mathrm{out}}(b, j) = w$, then $\ell^M_{\mathrm{w}}(w) = B_j$.

Informally, (i, b) represents the ith input of box b, (b, j) represents the jth output of box b, and the bijections θ^M_{in} and θ^M_{out} determine which wires are attached to which inputs and outputs, respectively. The labeling functions assign an object variable to each wire and a morphism variable to each box, and the typing conditions ensure that the sort of each wire matches the sort of each box it is attached to.

Definition 5.3 (Isomorphism) An *isomorphism* of diagrams $\psi : N \to M$ is given by a pair of bijections $\psi : W^N \to W^M$ and $\phi : B^N \to B^M$, commuting with the labeling functions and with θ_{in} and θ_{out}. Explicitly, this means that for all $w \in W^N$, and $b \in B^N$, and all $i \leqslant |\mathrm{dom}(\ell^N_{\mathrm{b}}(b))|$ and $j \leqslant |\mathrm{cod}(\ell^N_{\mathrm{b}}(b))|$,

$$\ell_{\mathrm{w}}^M(\psi(w)) = \ell_{\mathrm{w}}^N(w), \tag{5.2}$$
$$\ell_{\mathrm{b}}^M(\phi(b)) = \ell_{\mathrm{b}}^N(b), \tag{5.3}$$
$$\theta_{\mathrm{in}}^M(i, \phi(b)) = \psi(\theta_{\mathrm{in}}^N(i, b)), \tag{5.4}$$
$$\theta_{\mathrm{out}}^M(\phi(b), j) = \psi(\theta_{\mathrm{out}}^N(b, j)). \tag{5.5}$$

Lemma 5.4 *In the definition of isomorphism, the condition that ψ is a bijection is redundant.*

Proof. The bijection $\phi : B^N \to B^M$ induces a bijection $\hat{\phi} : \mathrm{Inputs}^N \to \mathrm{Inputs}^M$, defined by $\hat{\phi}(b, j) = (\phi(b), j)$. Equation (5.5) is then equivalent to the commutativity of this diagram:

$$\begin{array}{ccc} \mathrm{Inputs}^N & \xrightarrow{\theta_{\mathrm{out}}^N} & W^N \\ {\scriptstyle \hat{\phi}} \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathrm{Inputs}^M & \xrightarrow{\theta_{\mathrm{out}}^M} & W^M. \end{array}$$

Since the top, bottom, and left arrows are bijections, so is the right arrow. □

5.2 Interpretation in finite dimensional Hilbert spaces

Definition 5.5 (Interpretation) Let Σ be a signature. An *interpretation* $[\![-]\!]$ of Σ in finite dimensional Hilbert spaces assigns to each object variable $A \in \mathrm{Obj}$ a finite dimensional Hilbert space $[\![A]\!]$, and to each morphism variable $f : \boldsymbol{A} \to \boldsymbol{B}$ a linear map $[\![f]\!] : [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!] \to [\![B_1]\!] \otimes \ldots \otimes [\![B_m]\!]$, such that for all f, $[\![f^\dagger]\!] = [\![f]\!]^\dagger$.

We sometimes write $[\![\boldsymbol{A}]\!]$ for $[\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!]$.

Definition 5.6 (Denotation) Given a signature Σ, a (closed) diagram N, and an interpretation $[\![-]\!]$. Fix a unitary basis Basis_A for each space $[\![A]\!]$. An *indexing* of N is a function $\phi \in \prod_{w \in W^N} \mathrm{Basis}_{\ell_{\mathrm{w}}^N(w)}$, i.e., a choice of a basis element $\phi(w) \in \mathrm{Basis}_{\ell_{\mathrm{w}}^N(w)}$ for every wire $w \in W^N$. The set of indexings is written Idx^N. Then to each pair of an indexing ϕ and a box b, we assign a *matrix entry*

$$b(\phi) = ([\![f]\!])^{\phi(\theta_{\mathrm{in}}^N(1,b)),\ \ldots,\ \phi(\theta_{\mathrm{in}}^N(n,b))}_{\phi(\theta_{\mathrm{out}}^N(b,1)),\ \ldots,\ \phi(\theta_{\mathrm{out}}^N(b,m))}, \tag{5.6}$$

where $f = \ell_{\mathrm{b}}^N(b) : A_1 \otimes \ldots \otimes A_m \to B_1 \otimes \ldots \otimes B_m$. As before, we have written $F^{x_1,\ldots,x_n}_{y_1,\ldots,y_m}$ for the inner product $\langle y_1 \otimes \ldots \otimes y_m \mid F(x_1 \otimes \ldots \otimes x_n)\rangle$. The *denotation* of N is a scalar $[\![N]\!]$ defined as follows:

$$[\![N]\!] = \sum_{\phi \in \mathrm{Idx}^N} \prod_{b \in B^N} b(\phi). \tag{5.7}$$

Remark 5.7 The definition of $[\![N]\!]$ is independent of the chosen bases. In fact, the formula for $[\![N]\!]$ is just the usual formula for the summation over internal indices. It is the same as equation (4.1), expressed in the general context.

5.3 The M-interpretation

We are not in a position to give a formal proof of relative completeness (Lemma 4.1). Fix a diagram M. We will define a particular interpretation $[\![-]\!]_M$, called the *M-interpretation*, with the property that $[\![N]\!]_M = [\![M]\!]_M$ if and only if N and M are isomorphic.

A family $\{\xi_1, \ldots, \xi_k\}$ of complex transcendental numbers is *algebraically independent* if for every polynomial p with rational coefficients, $p(\xi_1, \ldots, \xi_k, \bar{\xi}_1, \ldots, \bar{\xi}_k) = 0$ implies $p \equiv 0$. We choose a family of algebraically independent transcendental numbers $\{\xi_b \mid b \in B^M\}$.

For each object variable A, let W_A^M be the set of all wires of M that are labeled A, and for each morphism variable f, let B_f^M be the set of boxes of M that are labeled f. In symbols,

$$W_A^M = \{w \in W^M \mid \ell_{\mathrm{w}}^M(w) = A\},$$
$$B_f^M = \{b \in B^M \mid \ell_{\mathrm{b}}^M(b) = f\}.$$

Then the M-interpretation $[\![-]\!]_M$ is defined as follows. For each A, let $[\![A]\!]_M$ be a Hilbert space with basis W_A^M. Suppose $f : \boldsymbol{A} \to \boldsymbol{B}$ is a morphism variable, and consider some f-labeled box $b \in B_f^M$. We define a linear map $F_b : [\![\boldsymbol{A}]\!]_M \to [\![\boldsymbol{B}]\!]_M$ by its matrix entries

$$(F_b)^{w_1,\ldots,w_n}_{w'_1,\ldots,w'_m} = \begin{cases} \xi_b & \text{if } w_i = \theta_{\mathrm{in}}^M(i,b) \text{ and } w'_j = \theta_{\mathrm{out}}^M(b,j) \text{ for all } i, j, \\ 0 & \text{else,} \end{cases} \tag{5.8}$$

where $w_i \in W_{A_i}^M$ and $w'_j \in W_{B_j}^M$ range over basis vectors. Finally, we define

$$[\![f]\!]_M = \sum_{b \in B_f^M} F_b + \sum_{b \in B_{f^\dagger}^M} F_b^\dagger. \tag{5.9}$$

5.4 Completeness

We must prove that the M-interpretation satisfies relative completeness (Lemma 4.1). First, we compute the M-interpretation of any diagram N. By (5.7) and (5.6), we have

$$[\![N]\!]_M = \sum_{\phi \in \mathrm{Idx}^N} \prod_{b \in B^N} ([\![\ell_{\mathrm{b}}^N(b)]\!]_M)^{\phi(\theta_{\mathrm{in}}^N(1,b)),\ \ldots,\ \phi(\theta_{\mathrm{in}}^N(n,b))}_{\phi(\theta_{\mathrm{out}}^N(b,1)),\ \ldots,\ \phi(\theta_{\mathrm{out}}^N(b,m))}$$

Using (5.9), it follows that

$$\begin{aligned} [\![N]\!]_M = & \sum_{\phi \in \mathrm{Idx}^N} \prod_{b \in B^N} \sum_{b' \in B^M_{\ell_{\mathrm{b}}^N(b)}} (F_{b'})^{\phi(\theta_{\mathrm{in}}^N(1,b)),\ \ldots,\ \phi(\theta_{\mathrm{in}}^N(n,b))}_{\phi(\theta_{\mathrm{out}}^N(b,1)),\ \ldots,\ \phi(\theta_{\mathrm{out}}^N(b,m))} \\ & + \sum_{\phi \in \mathrm{Idx}^N} \prod_{b \in B^N} \sum_{b' \in B^M_{\ell_{\mathrm{b}}^N(b)^\dagger}} (F_{b'}^\dagger)^{\phi(\theta_{\mathrm{in}}^N(1,b)),\ \ldots,\ \phi(\theta_{\mathrm{in}}^N(n,b))}_{\phi(\theta_{\mathrm{out}}^N(b,1)),\ \ldots,\ \phi(\theta_{\mathrm{out}}^N(b,m))} \end{aligned}$$

Now, using (5.8), we obtain the following explicit summation formula:

$$\llbracket N\rrbracket_M = \sum_{\phi\in \mathrm{Idx}^N}\prod_{b\in B^N}\sum_{b'\in B^M_{\ell^N_{\mathrm{b}}(b)}}\begin{cases}\xi_{b'} & \text{if } \phi(\theta^N_{\mathrm{in}}(i,b)) = \theta^M_{\mathrm{in}}(i,b') \text{ and} \\ & \phi(\theta^N_{\mathrm{out}}(b,j)) = \theta^M_{\mathrm{out}}(b',j) \text{ for all } i,j, \\ 0 & \text{else,}\end{cases}$$
$$+\sum_{\phi\in \mathrm{Idx}^N}\prod_{b\in B^N}\sum_{b'\in B^M_{\ell^N_{\mathrm{b}}(b)^\dagger}}\begin{cases}\bar\xi_{b'} & \text{if } \phi(\theta^N_{\mathrm{out}}(b,i)) = \theta^M_{\mathrm{in}}(i,b') \text{ and} \\ & \phi(\theta^N_{\mathrm{in}}(j,b)) = \theta^M_{\mathrm{out}}(b',j) \text{ for all } i,j, \\ 0 & \text{else.}\end{cases}$$

We note at this point that $\llbracket N\rrbracket_M$ can be (uniquely) written as a polynomial with non-negative integer coefficients in the variables $\{\xi_b,\bar\xi_b \mid b\in B^M\}$. Moreover, this polynomial is homogeneous.

By definition, $b'\in B^M_{\ell^N_{\mathrm{b}}(b)}$ if and only if $\ell^M_{\mathrm{b}}(b') = \ell^N_{\mathrm{b}}(b)$, and $b'\in B^M_{\ell^N_{\mathrm{b}}(b)^\dagger}$ if and only if $\ell^M_{\mathrm{b}}(b') = \ell^N_{\mathrm{b}}(b)^\dagger$. The sets $B^M_{\ell^N_{\mathrm{b}}(b)}$ and $B^M_{\ell^N_{\mathrm{b}}(b)^\dagger}$ are disjoint for each given b, since $\dagger$ is fixed-point free. We can therefore rewrite the summation as:

$$\llbracket N\rrbracket_M = \sum_{\phi\in \mathrm{Idx}^N}\prod_{b\in B^N}\sum_{b'\in B^M}\begin{cases}\xi_{b'} & \text{if } \ell^M_{\mathrm{b}}(b') = \ell^N_{\mathrm{b}}(b) \text{ and} \\ & \phi(\theta^N_{\mathrm{in}}(i,b)) = \theta^M_{\mathrm{in}}(i,b') \text{ and} \\ & \phi(\theta^N_{\mathrm{out}}(b,j)) = \theta^M_{\mathrm{out}}(b',j) \text{ for all } i,j, \\ \bar\xi_{b'} & \text{if } \ell^M_{\mathrm{b}}(b') = \ell^N_{\mathrm{b}}(b)^\dagger \text{ and} \\ & \phi(\theta^N_{\mathrm{out}}(b,i)) = \theta^M_{\mathrm{in}}(i,b') \text{ and} \\ & \phi(\theta^N_{\mathrm{in}}(j,b)) = \theta^M_{\mathrm{out}}(b',j) \text{ for all } i,j, \\ 0 & \text{else.}\end{cases}$$

Finally, we use the distributive law to exchange the order of addition and multiplication.

$$\llbracket N\rrbracket_M = \sum_{\phi\in \mathrm{Idx}^N}\sum_{\psi:B^N\to B^M}\prod_{b\in B^N}\begin{cases}\xi_{\psi(b)} & \text{if } \ell^M_{\mathrm{b}}(\psi(b)) = \ell^N_{\mathrm{b}}(b) \text{ and} \\ & \phi(\theta^N_{\mathrm{in}}(i,b)) = \theta^M_{\mathrm{in}}(i,\psi(b)) \text{ and} \\ & \phi(\theta^N_{\mathrm{out}}(b,j)) = \theta^M_{\mathrm{out}}(\psi(b),j) \text{ for} \\ & \text{all } i,j, \\ \bar\xi_{\psi(b)} & \text{if } \ell^M_{\mathrm{b}}(\psi(b)) = \ell^N_{\mathrm{b}}(b)^\dagger \text{ and} \\ & \phi(\theta^N_{\mathrm{out}}(b,i)) = \theta^M_{\mathrm{in}}(i,\psi(b)) \text{ and} \\ & \phi(\theta^N_{\mathrm{in}}(j,b)) = \theta^M_{\mathrm{out}}(\psi(b),j) \text{ for} \\ & \text{all } i,j, \\ 0 & \text{else.}\end{cases} \tag{5.10}$$

Now consider a fixed $\phi\in\mathrm{Idx}^N$ and fixed $\psi : B^N\to B^M$. We claim that the product $\prod_{b\in B^N}(\ldots)$ in (5.10) is equal to $\prod_{b\in B^M}\xi_b$ if and only if the pair of maps (ϕ,ψ) forms an isomorphism of diagrams from N to M. Indeed, the product in question is equal to $\prod_{b\in B^M}\xi_b$ if and only if ψ is a bijection and the first side condition of (5.10),

$$\ell^M_{\mathrm{b}}(\psi(b)) = \ell^N_{\mathrm{b}}(b) \text{ and}$$
$$\phi(\theta^N_{\mathrm{in}}(i,b)) = \theta^M_{\mathrm{in}}(i,\psi(b)) \text{ and } \phi(\theta^N_{\mathrm{out}}(b,j)) = \theta^M_{\mathrm{out}}(\psi(b),j) \text{ for all } i,j,$$

is satisfied for all $b\in B^N$. This side condition amounts precisely to conditions (5.2), (5.4), and (5.5) in the definition of isomorphism. Moreover, the requirement that ϕ (viewed as a function from W^N to W^M) is an indexing is equivalent to condition (5.3). By Lemma 5.4, these conditions are necessary and sufficient for the pair (ϕ,ψ) to be an isomorphism of diagrams.

Proof of Lemma 4.1 We have already noted that the right-to-left direction is trivial. For the left-to-right direction, note that we just showed that the polynomial $\llbracket N\rrbracket_M$ has a non-zero coefficient at $\prod_{b\in B^M}\xi_b$ if and only if there exists an isomorphism between N and M. In particular, $\llbracket M\rrbracket_M$ has a non-zero such coefficient. Therefore, if $\llbracket N\rrbracket_M = \llbracket M\rrbracket_M$, then $\llbracket N\rrbracket_M$ has a non-zero such coefficient, and it follows that $N\cong M$. □

The proof of Lemma 4.1 yields a stronger property:

Corollary 5.8 *The coefficient of $\llbracket N\rrbracket_M$ at the monomial $\prod_{b\in B^M}\xi_b$ is equal to the number of different isomorphisms from N to M.*

By the reductions of Section 3, this also proves Theorem 2.2.

6 Generalizations

6.1 Other fields

The result of this paper (Theorem 2.2) can be adapted to other fields besides the complex numbers. It is true for any field k of characteristic 0 with a non-trivial involutive automorphism $x\mapsto\bar x$. (Non-trivial means that for some x, $\bar x\neq x$).

The only special property of $\mathbb{C}$ that was used in the proof, and which may not hold in a general field k, was the existence of transcendentals. This problem is easily solved by first considering the field of fractions $k(x_1,\ldots,x_n)$, where the required transcendentals have been added freely. The proof of Lemma 4.1 then proceeds without change. Finally, once an interpretation over $k(x_1,\ldots,x_n)$ has been found such that $\llbracket M\rrbracket\neq\llbracket N\rrbracket$, we use the fact that in a field of characteristic 0, any non-zero polynomial has a non-root. Thus we can instantiate $x_1,\ldots,x_n$ to specific elements of k while preserving the inequality $\llbracket M\rrbracket\neq\llbracket N\rrbracket$. Note that therefore, Theorem 2.2 holds for k; however, Lemma 4.1 only holds for $k(x_1,\ldots,x_n)$.

6.2 Bounded dimension

The interpretation $\llbracket - \rrbracket_M$ constructed in Section 4.2 uses Hilbert spaces of unbounded dimension. One may ask whether Theorem 2.2 remains true if the dimension of the Hilbert spaces is fixed to some n. This is known to be false when $n = 2$. Here is a counterexample due to Bob Paré: the equation $\mathrm{tr}(AABBAB) = \mathrm{tr}(AABABB)$ holds for all 2×2-matrices, but does not hold in the graphical language. Indeed, by the Cayley-Hamilton theorem, $A^2 = \mu A + \nu I$ for some scalars μ,ν. Therefore

$$\mathrm{tr}(AABBAB) = \mu\,\mathrm{tr}(ABBAB) + \nu\,\mathrm{tr}(BBAB),$$
$$\mathrm{tr}(AABABB) = \mu\,\mathrm{tr}(ABABB) + \nu\,\mathrm{tr}(BABB),$$

and the right-hand-sides are equal by cyclicity of trace. It is not currently known to the author whether Theorem 2.2 is true when restricted to spaces of dimension 3.

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References

[1] Abramsky, S. and B. Coecke, *A categorical semantics of quantum protocols*, in: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, LICS 2004*, IEEE Computer Society Press, 2004, pp. 415–425.

[2] Hasegawa, M., M. Hofmann and G. Plotkin, *Finite dimensional vector spaces are complete for traced symmetric monoidal categories*, in: *Pillars of Computer Science: Essays Dedicated to Boris (Boaz) Trakhtenbrot on the Occasion of His 85th Birthday*, Lecture Notes in Computer Science 4800 (2008), pp. 367–385.

[3] Joyal, A., R. Street and D. Verity, *Traced monoidal categories*, Mathematical Proceedings of the Cambridge Philosophical Society **119** (1996), pp. 447–468.

[4] Selinger, P., *Dagger compact closed categories and completely positive maps*, in: *Proceedings of the 3rd International Workshop on Quantum Programming Languages*, Electronic Notes in Theoretical Computer Science 170 (2007), pp. 139–163.