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The main result of [2], i.e. if $T$ is a theory with strong iterate then the theory of reduced $\Sigma$-flowcharts $RFl_{\Sigma,T}$ is the theory with strong iterate freely generated by adding $\Sigma$ to $T$, was proved only when $T$ is an "almost syntactical" theory. Here we show that this technical condition is superfluous.

We shall use the notations from [2]. Suppose $T$ is an S-sorted theory with strong iterate and $\Sigma$ a double ranked S-sorted set. If one re-reads [2] he can see that in the definition of a reduction $f \underset{y}{\longmapsto} f'$ the condition wac): $Ac(f) \subseteq Dom(f)$ was wrote in this way only for the sake of generality. In fact he can use only reductions $f \underset{y}{\longmapsto} f'$ which fulfils a strong condition <u>sac): Dom(y) fulfils ac) in f</u> .

We shall distinguish two types of reductions, one by functions, denoted by $\underset{y}{\longrightarrow}$ and one by injective partial functions, denoted by $\underset{y}{\vdash\!\!\!-\!\!\!-}$ . The reductions by bijective functions are of both types. Obviously $\longmapsto \; \subseteq \; \vdash\!\!\!-\!\!\!- \circ \longrightarrow$.

Somehow as in the proof of Church-Rosser theorem [1] we shall show that $\longrightarrow$-reductions can pass over $\vdash\!\!\!-\!\!\!-$-reductions.

<u>Lemma 1.</u> If $f \underset{y_1}{\vdash\!\!\!-\!\!\!-} \tilde{f} \underset{y}{\longrightarrow} f'$ then there exist $\bar{f}$, $z_1$, $z_2$ such that $f \underset{z_1}{\longrightarrow} \bar{f} \underset{z_2}{\vdash\!\!\!-\!\!\!-} f'$.

<u>Proof.</u> Suppose the flowcharts are from $a$ to $b$ and $f = (i,t,e)$ with $e = e_1 \ldots e_{|e|}$ . Using a top reduction given by a bijective function we can restrict our analysis to the case when $[\![ Dom(y_1) ]\!] = \{1,\ldots,k\}$, for one $k \leq |e|$ . Now $y_1 = (1_{e_1 \ldots e_k} + \perp_{e_{k+1} \ldots e_{|e|}}) \cdot u$ for one bijective function $u$ , which can be moved in $y$ . Hence we can suppose

$y_1 = 1_{e_1\dots e_k} + \bot_{e_{k+1}\dots e_{|e|}}$ .

The $\bar{f}$-flowchart is obtained by adding to $f'$ an inaccessible copy of the $e_{k+1}\dots e_{|e|}$-part of $f$ . More precisely $\bar{f} = (\bar{i},\bar{t},\bar{e})$ is given by

$$\bar{e} = e'e_{k+1}\dots e_{|e|} \ ,$$

$$\bar{i} = i'(1_{p'} + 0_s + 1_b) \ ,$$

$$\bar{t} = \langle t'(1_{p'}+0_s+1_b) \ , \ (0_{e_1\dots e_k} + 1_{e_{k+1}\dots e_{|e|}})_{out} \ t \ (y_{in} + 1_{sb})\rangle$$

where $s = r^{*}_{in}(e_{k+1}\dots e_{|e|})$. Now we shall show that

$$f \xrightarrow[y+1_{e_{k+1}\dots e_{|e|}}]{} \bar{f} \ \underset{1_{e'}+\bot_{e_{k+1}\dots e_{|e|}}}{\vdash\!\!\!\relbar\!\!\!\relbar\!\!\!\relbar} \ f' \ .$$

One can easy see that $1_{e'}+\bot_{e_{k+1}\dots e_{|e|}}$ gives a reduction of desir[redacted]d type. A bit more difficult is to show that the total surjective function $y + 1_{e_{k+1}\dots e_{|e|}}$ reduces $f$ to $\bar{f}$ . By our hypothesis

$$\bar{i} = i'(1_{p'}+0_s+1_b) = i((y_1 y)_{in}+1_b)(1_{p'}+0_s+1_b) = i(y+\bot_s 0_s+1_b) =$$

$$= i(y+1_s+1_b)$$

where the last equality is based on $Im(i) \subseteq Dom(y_1)$. If $j\in[k]$ then as $[\![ Dom(y_1)]\!] = \{1,\dots,k\}$ fulfils ac) in $f$ , we have

$$\bar{t}_{y(j)} = t'_{y(j)}(1_{p'}+0_s+1_b) = t_j(y+\bot_s 0_s+1_b) = t_j(y+1_s+1_b).$$

In the case $j\in\{k+1,\dots,|e|\}$ we have even an identity. $\square$

<u>Lemma 2.</u> Every chain $f = f^0 \underset{y_1}{\longmapsto} f^1 \underset{y_2}{\longmapsto} \dots \underset{y_n}{\longmapsto} f^n = f'$ may be replaced with a two-step reduction $f \longrightarrow f'' \vdash\!\!\!\relbar\!\!\!\relbar f'$ (hence we have $\overset{*}{\longmapsto} = \longrightarrow \circ \vdash\!\!\!\relbar\!\!\!\relbar$ ).

<u>Proof.</u> By lemma 1 we can suppose that in $f^0 \underset{y_1}{\longmapsto} \dots \underset{y_n}{\longmapsto} f^n$ the first $k$ reductions are $\longrightarrow$-reductions and the last $n-k$ reductions are $\vdash\!\!\!\relbar\!\!\!\relbar$-reductions. An easy computation shows that every chain of

$\longrightarrow$-reductions can be replaced with one $\longrightarrow$-reduction. The same is true for $\longmapsfrom$ -reductions because if $\bar{f} \vdash_y \tilde{f}$ then

$$[\![x]\!] \text{ fulfils ac) in } \bar{f} \quad \text{iff} \quad y([\![x]\!]) \text{ fulfils ac) in } \tilde{f}$$

(the reverse implication is based on $[\![\mathrm{Im}(gz)]\!] = z([\![\mathrm{Im}(g)]\!])$ , if $z$ is an injective partial function). □

Lemma 3. If for $f : a \to ac$, $f' : b \to bc$ and $y \in \mathrm{Str}(a,b)$ surjective function, we have $f(y+1_c) \xrightarrow[z]{} yf'$ then $f^{+} \xrightarrow[z]{} y(f')^{+}$ .

Proof. See the last part of the proof of theorem 11.2 in [2] . □

Theorem 4. If $T$ is a theory with strong iterate then $RFl_{\Sigma,T}$ is a theory with strong iterate.

Proof. We have only to show I4-S :

if $f;a \to ac$, $f':b \to bc$ and $y \in \mathrm{Str}(a,b)$ is a surjective function such that $f(y+1_c) \equiv yf'$ then $f^{+} \equiv y(f')^{+}$ .

Let us suppose that $f$, $f'$ are minimal flowcharts. Then $yf'$ is also a minimal flowchart. Hence the equivalence $f(y+1_c) \equiv yf'$ is, in fact, given by a chain of reductions $f(y+1_c) \longmapsto \ldots \longmapsto yf'$ . By lemma 2 this chain can be replaced with $f(y+1_c) \xrightarrow[z]{} f'' \vdash_u yf'$. As $i''(u_{in}+1_{bc}) = yi'$ and $u$ has a right inverse $v$ with $uv = \mathrm{Dom}(u)$ and $\mathrm{Im}(i'') \subseteq \mathrm{Dom}(u)_{in}+1_{bc}$ we have

$$i'' = i''(\mathrm{Dom}(u)_{in}+1_{bc}) = i''(u_{in}+1_{bc})(v_{in}+1_{bc}) = yi'(v_{in}+1_{bc}),$$

which shows that $f'' = y\bar{f}$ . By lemma 3, $f^{+} \longrightarrow y\bar{f}^{+}$. The theorem is concluded if we show that $\bar{f}^{+} \longmapsto (f')^{+}$. By lemma 10.1 in [2] , $y\bar{f} \vdash_u yf'$ implies $\bar{f} = wy\bar{f} \longmapsto wyf' = f'$ , where $w$ is a left inverse of $y$ $(wy=1_b)$ and $\bar{f}^{+} \longmapsto (f')^{+}$ . □

As a corollary we rewrite the main theorem from [2].

Main theorem. If $T$ is an $S$-sorted theory with strong iterate and $\Sigma$ a double ranked $S$-sorted set, then the theory of reduced flowcharts $RFl_{\Sigma,T}$ is the theory with strong iterate freely generated by adding $\Sigma$ to $T$. $\Box$

By lemma 2, $\overset{*}{\longmapsto} = \longrightarrow \circ \longmapsfrom$. By Lemma 11.1 in [2], if $T$ is an almost syntactical theory then $\overset{*}{\longmapsto} = \longmapsto$. We conclude this note with an example, in the one-sorted case, which shows that generally $\overset{*}{\longmapsto} \neq \longmapsto$. As $T$ we chose the quotient of the theory of rational $\Gamma$-trees, $RT_\Gamma$ ($\Gamma$ is a one ranked set, with one $\gamma$ of arity two) by the congruence $\equiv$ generated by

$$\gamma < x_1^1, x_1^1 > = \perp_1 0_1 .$$

Now it is easy to see that $RT_\Gamma/\equiv$ is, as $RT_\Gamma$, a theory with strong iterate. Then the flowchart

$$(\gamma(<x_1^2, x_2^2> + 0_1) , \; 0_2 + 1_1 , \; \sigma\sigma)$$

where $r_{in}(\sigma) = r_{out}(\sigma) = 1$, can be reduced (in two steps) to $(\perp_1 0_1, 0_1, \lambda)$:

$$(\gamma(<x_1^1, x_1^1> + 0_1), 0_2 + 1_1, \sigma\sigma) \xrightarrow[<1_1,1_1>]{} (\perp_1 0_1 + 0_1, 0_1 + 1_1, \sigma) \underset{\perp_1}{\longmapsfrom} (\perp_1 0_1, 0_1, \lambda)$$

but not in a single step.

## References.

[1] J.R.HINDLEY, B.LERCHER and J.P.SELDIN, "Introduction to Combinatory Logic", Cambridge University Press, 1972.

[2] GH.ŞTEFĂNESCU, "On flowchart theories (I)", Preprint INCREST, No.39, 1984.