An Introduction to Finitesimal Topology*

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1 Motivation

Since students—especially undergraduates—usually experience considerable difficulties when first confronted with the complex notions of infinite topological spaces, the authors hope that this humble exposition will serve as a feasible introduction to the field.

To that end we confine ourselfes to the treatment of *finite* spaces, which allows us to exhibit the unimpaired beauty of the theory without unnecessarily having to descend into the depths of classical topology.

Of course we assume some familiarity with basic concepts like the notions of a *topological space* and a *Hausdorff space*, as well as basic knowledge of the *indiscrete* and *discrete topology* and of *automorphisms*. Presumably these have been covered in high school.

2 Introductory Example

To us all the most natural space probably appears to be the *evacuated space* $X = \emptyset$ with the natural topology $\mathcal{O} = \{\emptyset\}$. And in fact there exists no other topology on X (exercise!). Hence on the set X there is exactly one topology.

Let us now examine some more complicated spaces:

Exercise 2.1 *Show that on all one element spaces there is also exactly one topology (which one?)!*

Theorem 2.2 *There is no set, such that there exist exactly 2 topologies on it.*
Proof: Sets with zero or one elements have already been treated. Hence let X have at least two different elements a and b. In any event there is the indiscrete topology $\mathcal{I}$ and the discrete topology $\mathcal{D}$. These are different (why?). Consider further the topology $\mathcal{O} = \{X, \emptyset, \{a\}\}$ on X. It is readily seen that this is a topology. Because $\emptyset \neq \{a\} \neq X$ and $\{b\} \notin \mathcal{O}$, this topology is properly different from $\mathcal{I}$ and $\mathcal{D}$. $\square$

This result is easily strengthened to:

Theorem 2.3 *2 is the smallest positive integer for which Theorem 2.2 holds.*
Proof: Exercise! $\square$

*Originally "Einführung in die Finitesimaltopologie"

3 Concepts and Notation

No serious mathematical discourse is possible without introducing precise concepts and notation. Hence we introduce the following new notion:

Definiton 3.1 *A positive integer n is called* contratoponumatic *in case there is no set X such that there exist exactly n pairwise distinct topologies on X.*

Corollary 3.2 *2 is contratoponumatic.*
Proof: This is an immediate consequence of Theorem 2.2 and Definition 3.1. □

Remark 3.3 *The first contratoponumatic numbers are 2,3,5,6,7,8,9,10.*

One's first guess will probably be that all numbers $n \geq 5$ are contratoponumatic. But this is not the case. In fact, we even have the following

Theorem 3.4 *There is no number m, such that all $n \geq m$ are contratoponumatic.*
Proof: Let m be an arbitrarily chosen, but fixed, number. Now consider $X = \{1, 2, \ldots, m+1\}$ and for every $i \in \{1, 2, \ldots, m+1\}$ the topology $\mathcal{O}_i = \{\emptyset, X, \{i\}\}$. Obviously $i \neq j \Longrightarrow \mathcal{O}_i \neq \mathcal{O}_j$. Therefore there must be a number $n > m$ which is not contratoponumatic. □

In the proof of the preceding theorem we constructed a number n which was the total number of topologies on X. This stimulates the following definition.

Definiton 3.5 *Let X be a finite set. The* polytoponumatic character *of X is the number of distinct topologies on X. It is denoted by $\wp(X)$ (pronounced "gimmick of X"). Correspondingly the number of non-Hausdorff topologies on X is denoted $\wp_H(X)$, and we analogously call this number the* monotoponumatic character.

Exercise 3.6 *Show $\wp_H(X) = \wp(X) - 1$*
Hint: First consider $\wp(X) - \wp_H(X)$ and use the fact that X is finite!

4 Morphisms

The structure of mathematical theories becomes especially appreciable when structure preserving maps between the objects of the theory are considered.

Definiton 4.1 *The* polyautomorphity *of a finite topological space X* [1] *is the number of distinct automorphisms of X.*

Now we are in a position to formulate the following central theorem:

[1] Such a space is sometimes also called a *finitesimally topological space.*

Theorem 4.2 (FUNDAMENTAL THEOREM OF FINITESIMAL TOPOLOGY) *A number n is the polyautomorphity of some topological space X if and only if either n is contratoponumatic or $n = \wp(Y)$ for some space Y.*

Proof:

1. Let X be a finite topological space with polyautomorphity n. We first remark that the identity map $id_X : X \to X : x \mapsto x$ is an automorphism of X (why?). Hence $n \geq 1$. In case n is contratoponumatic, nothing is left to show. Otherwise by Definition 3.1 there is a set Y with exactly n topologies existing on it. The reader will easily check that this already has $\wp(Y) = n$ as a consequence.

2. Conversely, let n be contratoponumatic. We will then construct a topological space X with polyautomorphity n as follows: Let

$$X = \{i_0, j_0, k_0, i_1, j_1, k_1, \ldots, i_{n-1}, j_{n-1}, k_{n-1}\}$$

$$\mathcal{B} = \bigcup_{p=0}^{n-1} \{\{i_p\}, \{i_p, j_p\}, \{i_p, j_p, k_p, i_{p+1}\}\},$$

where the indices are calculated modulo n. $\mathcal{B}$ is a basis for a topology (why?). Let $\mathcal{O}$ be the topology generated by $\mathcal{B}$.

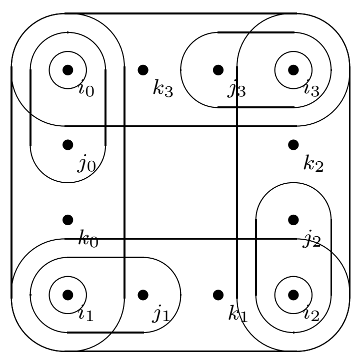

Figure 1: Basis for a topology with polyautomorphity 4

Then $\kappa : i_r \mapsto i_{r+1}, j_r \mapsto j_{r+1}, k_r \mapsto k_{r+1}$ is surely an automorphism of X.

Now let $\varphi : X \to X$ be an arbitrary automorphism of X. We will show that φ is of the form κ^p, for some $p \in \{0, \ldots, n-1\}$. If

$$\mathcal{H}(x) := \bigcap \{U \in \mathcal{O} | x \in U\}$$

denotes the smallest open set containing x, then we have $|\mathcal{H}(i_p)| = 1$, $|\mathcal{H}(j_p)| = 2$ and $|\mathcal{H}(k_p)| = 4$ for $p \in \{0, \ldots, n-1\}$. Hence the sets $\{i_0, \ldots, i_{n-1}\}$, $\{j_0, \ldots, j_{n-1}\}$ and $\{k_0, \ldots, k_{n-1}\}$ are φ-invariant.

Consequently, let $\varphi(k_0) = k_p$. From this we obtain:

$$\varphi(\{i_0, j_0, k_0, i_1\}) = \varphi(\mathcal{H}(k_0)) = \mathcal{H}(\varphi(k_0)) = \mathcal{H}(k_p) = \{i_p, j_p, k_p, i_{p+1}\}$$

(Why do $\mathcal{H}$ and φ commute?) In particular $\varphi(j_0) = j_p$. *A priori* there are the two possibilities $\varphi(i_0) = i_p$ or $\varphi(i_0) = i_{p+1}$. But

$$\varphi(\{i_0, j_0\}) = \varphi(\mathcal{H}(j_0)) = \mathcal{H}(\varphi(j_0)) = \mathcal{H}(j_p)) = \{i_p, j_p\},$$

whence $\varphi(i_0) = i_p$. By continuing this argument for $\varphi(\{i_1, j_1, k_1, i_2\})$ etc. we obtain $\varphi = \kappa^p$. Hence on X there are exactly the n automorphisms $\kappa^0, \kappa^1, \ldots, \kappa^{n-1}$. But this means n is the polyautomorphity of X.

3. A similar argument works in the case when n is a polytoponumatic character.

□

The relevance of the following statement, which we present to the reader as a little exercise, is essential.

Exercise 4.3 *Show that for any polyautomorphity n of some topological space X there is either a set Y with $\wp(Y) = n$ or n is contratoponumatic.*

5 Continuity

Considerations of mappings from a finitesimally topological space X into an arbitrary topological space Y naturally raise the question whether such a mapping is continuous.

Therefore we want to start by collecting some basic facts.

Lemma 5.1 *The image of a finitesimally topological space X under an* arbitrary *map f is finite.*
Proof: (easy) exercise! □

This yields the following central

Theorem 5.2 *If under the hypotheses of Lemma 5.1 $A \subseteq X$ is an arbitrary subset of X, then $f(A)$ is finite as well.*
Proof: If $\mathcal{O}$ is the topology of X, then $(A, \mathcal{O}|A)$, the topological subspace A with the subspace topology, is also finitesimally topological. With Lemma 5.1, applied to $(A, \mathcal{O}|A)$ and $f|A$, the claim follows. □

Of particular interest to us are—of course—Hausdorff spaces. The following theorem gives us substantial insight.

Theorem 5.3 (Continuation Theorem) *Let X be a finite topological Hausdorff space, $A \subseteq X$ a subspace and $Y \neq \emptyset$ an arbitrary topological space. Then any continuous function $f : A \to Y$ has a continuation $\tilde{f} : X \to Y$ with $\tilde{f}|A = f$, where $\tilde{f}$ is continuous on all of X*
Proof: Since $Y \neq \emptyset$, there is some $y \in Y$. One defines

$$\tilde{f} : X \to Y : x \mapsto \begin{cases} f(x) & \text{if} \quad x \in A \\ y & \text{if} \quad x \notin A \end{cases}$$

Obviously $\tilde{f}|A = f$. Only the continuity of $\tilde{f}$ remains to be shown: Let $U \subseteq Y$ be open. Let $x \in \tilde{f}^{-1}(U) =: V$ be arbitrary, but fixed. Now consider $W := \mathcal{H}(x)$ (see the proof of Theorem 4.2). W is open, since it is a finite intersection of open sets. Note that $W \subseteq V$, because otherwise $W \setminus V$ would be non-empty, in which case there would be a $w \in W \setminus V$. By Definiton of $\mathcal{H}$, w is contained in every open neighborhood of x. The Hausdorff condition implies $w = x$, contradicting $x \in V$. Therefore $x \in W \subseteq V$ and V is a neighborhood of all its points and hence open. But this implies the continuity of $\tilde{f}$. $\square$

We hope that out little delineation has lead the reader to a closer understanding of the basics of this extensive field. Unfortunately, for the sake of due brevity, a lot of material could not be included. Perhaps the interested reader will now apply these newly acquired finitesimally topological insights towards further, independent studies in topology. We maintain that if the reader has worked sufficiently hard on the exercises, he/she should by now be more than well prepared.