Combinatorics

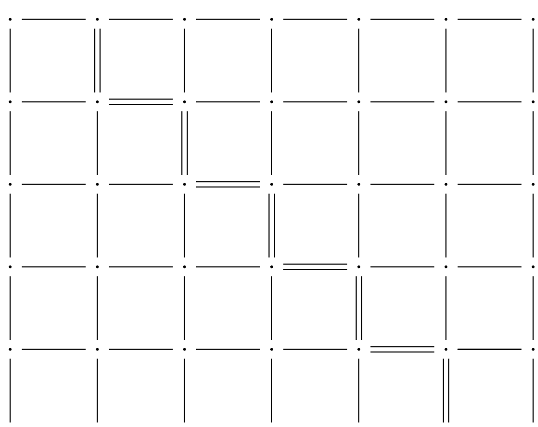
Covers various areas of discrete mathematics, such as counting, analysing various discrete structures, such as graphs, or partially-ordered sets. It also deals with problems about whether certain types of structures can or must exist — for example, if we colour all the natural numbers with two colours, must there be three numbers x, y and z such that $x + y = z$?

There are a number of different branches of combinatorics:

Ennumerative Combinatorics

This deals with counting. The basic principle of counting is the multiplication principle, which states that if for each of m possible starts of patterns, there are n possible completions, then the total number of patterns is mn. Geometrically, this corresponds to arranging the things to be counted into a rectangle. This can be used in many situations, for example, the count the number of different orders (permutations) of n items, we can divide according to the first item. There are n possibilities for this. Then, whichever the first item is, there are $n-1$ possibilities for the second item, $n-2$ for the third and so on, so the total number of permutations is $n!$.

The other trick is multiple counting, where we count each item a fixed number of times, and then we can just divide by that number of times. An example of this is triangular numbers. We can calculate the triangular numbers by arranging two copies of the triangle into a rectangle.



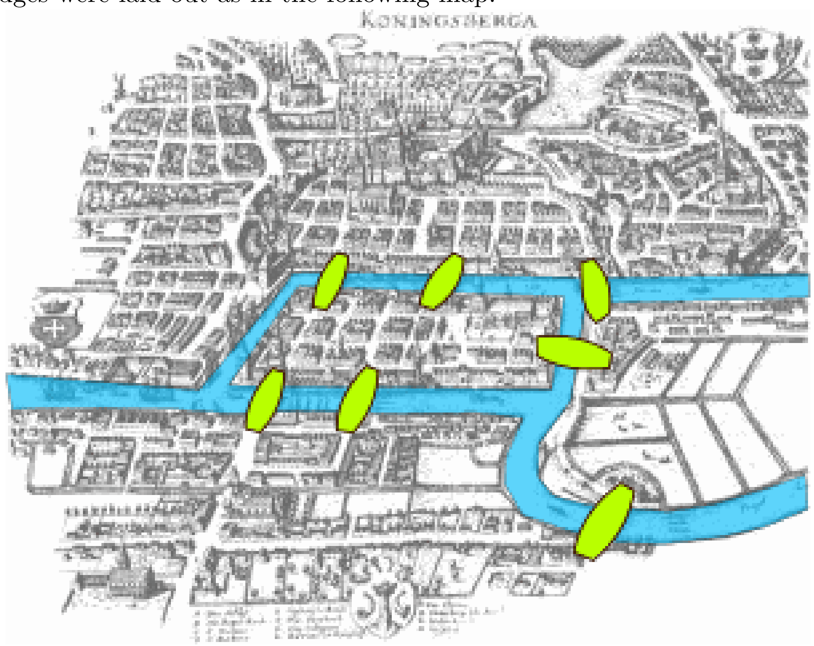
Graph Theory

A *graph* is a structure consisting of a collection of *vertices*, and *edges* joining pairs of vertices. Graphs can represent many different things, for example

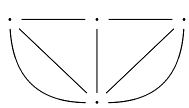
- vertices represent computers, edges represent network connections.
- vertices represent people, edges represent friendships.

- vertices represent places, edges represent routes between them.

The origins of graph theory are from Leonhard Euler's solution to the bridges of Königsberg problem, which asked whether it was possible to walk around Königsberg in such a way as to cross each of the 7 bridges exactly once. The bridges were laid out as in the following map:



Euler represented the important information from the map in the form of the graph, where the vertices represented the four land masses (two river banks and two islands) and the edges represented the bridges. The resulting graph was as follows:



[The left and right vertices represent the two banks, and the middle two vertices represent the islands.]

Euler then observed that in any walk, the number of edges leaving a given vertex, other than the starting and ending vertex, must be equal to the number of edges entering that vertex. Therefore, the number of edges at each vertex, except for the starting and ending vertices, must be even. However, in the above graph, there are an odd number of edges at each vertex, so it is not possible to find such a walk.

One of the most famous results in graph theory is the four-colour map theorem: Given a map drawn in the plane, the regions may be coloured using only four colours, in such a way that adjacent regions are coloured differently. This theorem is also philosophically interesting (i.e. controversial) because the proof involved the use of a computer to check a number of outstanding cases.

There are many simple-to state open problems involving graphs and efficient algorithms for solving certain graph-theoretical problems. For example, the *travelling salesman problem*, which considers a graph where all edges have lengths, and asks what is the length of the shortest cycle that passes through every vertex, is one of the best-known NP-hard problems. Even the problem of deciding whether two graphs are effectively the same is a difficult problem.

Ramsey Theory, Extremal Combinatorics

These areas deal with questions of how large a structure can be subject to certain properties. For example, "How many edges can a graph have without containing a triangle (three vertices such that all three edges between them are in the graph)?"

Ramsey theory studies when certain patterns must occur somewhere within a structure of a given size. The most basic example is the pigeon-hole principle, which states that if we have n items, and we divide them into $n-1$ (or fewer categories), then there must be two items in some category. For example, given any 13 people, there must be two of them whose birthday is in the same month. This principle turns out to be surprisingly effective for a number of combinatorial problems in mathematics.

The basic result that leads in to Ramsey theory is the following: "If we take any 6 people, there must either be some three of them who all know each other, or who are all strangers." Mathematically, we model this statement by drawing a graph with people as vertices, and one edge between any pair of people. Then we represent two people knowing one another by colouring the edge between them with one colour, and represent their being strangers by colouring the edge a different colour. Then the mathematical result becomes "If we colour the edges of a complete graph on 6 vertices with 2 colours, then there are always 3 vertices such that all edges between them are the same colour." This result can be extended to larger numbers. That is, we can always find a number N large enough that if we colour the edges of a complete graph on N vertices with m colours, there will be some n vertices such that all edges between them are the same colour. The smallest possible value of this N is still unknown in many cases.

Combinatorial Geometry

There are a number of combinatorial questions in geometry, which ask whether certain configurations are possible. For example, "Is it possible for 13 spheres of radius 1 to all touch a fixed sphere of radius 1 without overlapping?" "What convex polyhedra and higher dimensional polytopes exist?"