MATH 1115, Mathematics for Commerce
WINTER 2011
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Homework Sheet 7
Model Solutions

Each multiple choice question is worth one mark, other questions are worth two marks. Show your working for the other questions, but for multiple choice questions, just the letter is sufficient.

1. *Calculate* $\lim_{x\to 2} x^2+7$.

   This is a polynomial function, so it is continuous. Therefore $\lim_{x\to 2} x^2+7 = 2^2+7 = 11$.

   (A) 7

   (B) 9

   (C) 11

   (D) ∞

   (E) The limit does not exist and is not ∞

2. *Calculate* $\lim_{x\to 2} \frac{(x-2)^3(x+3)(x-5)}{(x-2)^2(x+6)}$.

   When $x \neq 2$, $\frac{(x-2)^3(x+3)(x-5)}{(x-2)^2(x+6)} = \frac{(x-2)(x+3)(x-5)}{(x+6)}$. This is a quotient of continuous functions $(x-2)(x+3)(x-5)$ and $x+6$. The latter is not 0 at $x=2$, so we have $\lim_{x\to 2} \frac{(x-2)^3(x+3)(x-5)}{(x-2)^2(x+6)} = \lim_{x\to 2} \frac{(x-2)(x+3)(x-5)}{x+6} = \frac{\lim_{x\to 2}(x-2)(x+3)(x-5)}{\lim_{x\to 2}(x+6)} = \frac{0}{8} = 0$

   (A) $-\frac{15}{8}$

   (B) 0

   (C) 15

   (D) ∞

   (E) The limit does not exist and is not ∞

3. *Calculate* $\lim_{x\to\infty} \frac{3x^3+2x^2-4x+1}{2x^3-4x^2+3x-7}$

   This is a rational function, and a limit as $x\to\infty$, so it can be found by taking just the highest-order terms, that is

$$\lim_{x\to\infty} \frac{3x^3+2x^2-4x+1}{2x^3-4x^2+3x-7} = \lim_{x\to\infty} \frac{3x^3}{2x^3} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}$$

More formally, if we let $z = \frac{1}{x}$, then

$$\lim_{x\to\infty} \frac{3x^3 + 2x^2 - 4x + 1}{2x^3 - 4x^2 + 3x - 7} = \lim_{z\to 0} \frac{\frac{3}{z^3} + \frac{2}{z^2} - \frac{4}{z} + 1}{\frac{2}{z^3} - \frac{4}{z^2} + \frac{3}{z} - 7} = \lim_{z\to 0} \frac{3 + 2z - 4z^2 + z^3}{2 - 4z + 3z^2 - 7z^3} = \frac{3}{2}$$

(A) $-\frac{1}{2}$

(B) 0

(C) $\mathbf{\frac{3}{2}}$

(D) ∞

(E) The limit does not exist and is not ∞

4. *Which of the following functions is continuous at $x = 3$?*

(A) $\mathbf{f(x)} = \begin{cases} x^2 - 4 & \textbf{if } x \leqslant 3 \\ x + 2 & \textbf{if } x > 3 \end{cases}$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x + 2 = 3 + 2 = 5$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 4 = 9 - 4 = 5$$

$f(3) = 3^2 - 4 = 5$

(B) $f(x) = \frac{x^3+7}{x-3}$

$$\lim_{x\to 3^+} f(x) = \infty$$

So f does not have a limit as $x \to 3$. (Also $f(3)$ is undefined.)

(C) $f(x) = \begin{cases} x - 2 & \text{if } x \leqslant 3 \\ x + 2 & \text{if } x > 3 \end{cases}$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x + 2 = 3 + 2 = 5$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x - 2 = 3 - 2 = 1$$

(D) $f(x) = \begin{cases} x^2 - 2 & \text{if } x < 3 \\ 6 & \text{if } x = 3 \\ x + 4 & \text{if } x > 3 \end{cases}$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x + 4 = 3 + 4 = 7$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 2 = 9 - 2 = 7$$

$$f(3) = 6$$

So f has a limit as $x \to 3$, but the limit is not equal to $f(3)$.

(E) More than one of them is continuous at $x = 3$.

5. *Define $f(x)$ to be the fractional part of x, that is, the smallest non-negative number such that $x - f(x)$ is an integer $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. For example, $f(3) = 0$, $f(4.6) = 0.6$, $f(-2.4) = 0.6$.*

   Calculate $\lim\limits_{x\to\infty} f(x)$.

   To say that a limit exists means that as x gets larger, $f(x)$ gives a better approximation to the limit. However, no matter how large we take x, there will always be values of x such that $f(x) = 0$, and there will always be values such that $f(x) = 0.5$. These cannot both be good approximations to the limit, so the limit can't exist. The limit is not ∞ because $f(x)$ does not get very large (in fact it never gets larger than 1).

   (A) 0

   (B) $\frac{1}{2}$

   (C) 1

   (D) ∞

   (E) The limit does not exist and is not ∞

6. *The function $f(x) = \begin{cases} \frac{x^2-2}{(x-2)(x-4)} & \text{if } x \leqslant 3 \\ \frac{x+4}{x-1} & \text{if } x > 3 \end{cases}$ is continuous:*

   The part on $x \leqslant 3$ is a quotient of continuous functions, so is continuous except when the denominator is 0, which happens only at $x = 2$ (this part is only for $x \leqslant 3$, so $x = 4$ doesn't count). At $x = 2$, the denominator is 0 and the numerator is not 0, so the limit at x=2 does not exist. Therefore, f is discontinuous at $x = 2$. The part on $x > 3$ is also a quotient of continous functions, and the denominator is never 0 for $x > 3$, so f is continuous at all $x > 3$. The only remaining question is whether f is continuous at 3. Both parts are continuous at $x = 3$, so we just need to compare the limits:

   $\lim_{x\to 3^-} f(x) = \frac{3^2-2}{(3-2)(3-4)} = -7$, while $\lim_{x\to 3^+} f(x) = \frac{3+4}{3-1} = \frac{7}{2}$, so f is discontinuous at 3.

   (A) everywhere.

   (B) everywhere except at $x = 3$.

   (C) everywhere except at $x = 2$.

   (D) everywhere except at $x = 2$, $x = 3$, $x = 4$.

   (E) everywhere except at $\mathbf{x = 2}$ and $\mathbf{x = 3}$.

7. *Solve* $(x^2+4)(x-3)(x+2)^3(x-2) \leqslant 0$.

   We start by finding all solutions to $(x^2+4)(x-3)(x+2)^3(x-2) = 0$. Solutions occur when $x = 3$, $x = -2$, and $x = 2$. Therefore, on each of the intervals $(-\infty, -2)$, $(-2, 2)$ $(2, 3)$ and $(3, \infty)$, $(x^2+4)(x-3)(x+2)^3(x-2)$ is either always positive or always negative.

   solution 1: We try values in these intervals.

x	$(x^2+4)(x-3)(x+2)^3(x-2)$
-3	$13 \times (-6) \times (-1)^3 \times (-5) < 0$
0	$4 \times (-3) \times 2^3 \times (-2) > 0$
2.5	$10.25 \times (-0.5) \times (4.5)^3 \times (0.5) < 0$
4	$20 \times 1 \times 6^3 \times 2 > 0$

   So the solution to $(x^2+4)(x-3)(x+2)^3(x-2) \leqslant 0$ is $x \leqslant -3$ or $2 \leqslant x \leqslant 3$.

   solution 2: We consider the signs of each component on each interval

	$(-\infty, -2)$	$(-2, 2)$	$(2, 3)$	$(3, \infty)$
(x^2+4)	+	+	+	+
$(x-3)$	-	-	-	+
$(x+2)^3$	-	+	+	+
$(x-2)$	-	-	+	+
$(x^2+4)(x-3)(x+2)^3(x-2)$	-	+	-	+

   So the solution to $(x^2+4)(x-3)(x+2)^3(x-2) \leqslant 0$ is $x \leqslant -2$ or $2 \leqslant x \leqslant 3$.

8. *Give an example of two functions f and g such that $\lim_{x\to 2} f(x)$ and $\lim_{x\to 2} g(x)$ do not exist, but such that $\lim_{x\to 2} f(x) + g(x)$ exists.*

   There are many different types of examples. The easiest is probably $f(x) = \frac{1}{x-2}$, $g(x) = \frac{-1}{x+2}$.

   To create a general example, take a function f such that $\lim_{x\to 2} f(x)$ does not exist, and a function h such that $\lim_{x\to 2} h(x)$ does exist, and define $g(x) = h(x) - f(x)$.