MATH 1115, Mathematics for Commerce
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Homework Sheet 9
Model Solutions

1. *The second derivative of* $2x^4 - 3x^2 + 7x + 2$ *is:*

   The first derivative is $8x^3 - 6x + 7$. The second derivative is $24x^2 - 6$.

   (A) $8x^3 - 6x + 7$

   (B) $(8x^3 - 6x + 7)^2$

   (C) $6x^2 - 3$

   (D) $\mathbf{24x^2 - 6}$

   (E) undefined

2. *The function* $f(x) = x^4 + 2x^3 - 7x + 4$ *is:*

   $f'(x) = 4x^3 + 6x^2 - 7$, so $f'(-1) = -4 + 6 - 7 = -5 < 0$ so f is decreasing at $x = -1$. $f'(1) = 4 + 6 - 7 = 3 > 0$, so f is increasing at $x = 1$.

   (A) increasing at both $x = -1$ and at $x = 1$.

   (B) increasing at $x = -1$ but decreasing at $x = 1$.

   (C) decreasing at $\mathbf{x = -1}$ but increasing at $\mathbf{x = 1}$.

   (D) decreasing at both $x = -1$ and at $x = 1$.

   (E) at a local extremum at one of the points $x = 1$ and $x = -1$.

3. *The critical value of the function* $f(x) = \frac{x^2-2x+1}{x+3}$ *at* $x = 1$ *is:*

   By the quotient rule,

   $f'(x) = \frac{(2x-2)(x+3)-(x^2-2x+1)}{(x+3)^2} = \frac{x^2+6x-7}{(x+3)^2} = \frac{(x-1)(x+7)}{(x+3)^2}$. We easily see that $f'(1) = 0$, so $x = 1$ is a critical value.

   solution 1:

   f' is differentiable at $x = 1$, so we can apply the second derivative test to determine whether $x = 1$ is a minimum or maximum.

   Again by the quotient rule,

   $f''(x) = \frac{(2x+6)(x+3)^2-(x-1)(x+7)2(x+3)}{(x+3)^4}$, so $f''(1) = \frac{8\times 4^2-0}{4^4} > 0$. Therefore, $x = 1$ is a local minimum for f.

   solution 2: It is easy to see that for $-3 < x < 1$, $(x-1)$ is negative, $(x+7)$ is positive, and $(x+3)$ is positive, so $f'(x)$ is negative. Similarly, for $x > 1$, $(x-1)$, $(x+7)$ and $(x+3)$ are all positive, so $f'(x)$ is positive. Therefore, $x = 1$ is a local minimum.

   However, we can see that for $x < -3$, $f(x)$ is negative, while $f(1) = 0$, so $x = 1$ is not a global minimum.

(A) a local (relative) maximum, but not a global (absolute) maximum.

(B) a global (absolute) maximum.

(C) a local (relative) minimum, but not a global (absolute) minimum.

(D) a global (absolute) minimum.

(E) Neither a local minimum nor a local maximum.

4. *Which of the following lines is an asymptote to the function* $f(x) = \frac{x^4-2x+3}{x^2+3}$*?*

The denominator $x^2 + 3$ is always positive, and in particular, never 0, so there are no vertical asymptotes.

By long division, $f(x) = x^2 - 3 - \frac{2x-12}{x^2+3}$, so we see that for any a and b, $\lim_{x\to\infty} f(x) - ax - b = \infty$, so $f(x)$ does not have a horizontal or vertical asymptote.

(A) $x = \sqrt{3}$

(B) $y = 3$

(C) $y = 2x + 3$

(D) None of these, but $f(x)$ does have an asymptote.

(E) f(x) does not have any asymptotes.

5. *A company's revenue, r as a function of the amount of production x is given by $r = 20x - 3\sqrt{x}$. Meanwhile the cost of production, c is given by $c = \frac{3x^2+13x}{x+1}$. The amount of production x which maximises the company's profit is a solution to (Assuming there is some x which maximises profit):*

[Bonus question: What do the the solutions to the wrong answers in Q. 5 (assuming they exist) correspond to?]

(A) $2x + 10\sqrt{x} = \frac{3x^2+13x}{x+1}$

This equation says that revenue is equal to cost, i.e. profit is 0.

(B) $2 + \frac{5}{\sqrt{x}} = 0$

This equation says that $\frac{dr}{dx} = 0$, so it gives the maximum or minimum revenue. (In fact it has no solutions because the revenue is an increasing function of x.)

(C) $\mathbf{2 + \frac{5}{\sqrt{x}} = \frac{3x^2+6x+13}{(x+1)^2}}$

This says $\frac{dr}{dx} = \frac{dc}{dx}$, so we get $\frac{dr}{dx} - \frac{dc}{dx} = 0$, so since profit p is given by $p = r - c$, this says $\frac{dp}{dx} = 0$, i.e. a local maximum or minimum for profit.

(D) $(2x + 10\sqrt{x})\left(\frac{3x^2+6x+13}{(x+1)^2}\right) = \left(2 + \frac{5}{\sqrt{x}}\right)\left(\frac{3x^2+13x}{x+1}\right)$

This says $r\frac{dc}{dx} = c\frac{dr}{dx}$, or equivalently $r\frac{dc}{dx} - c\frac{dr}{dx} = 0$, and dividing through by c^2, we get $\frac{r\frac{dc}{dx} - c\frac{dr}{dx}}{c^2} = 0$. This is the derivative of $\frac{r}{c}$. This being zero represents a maximum or minimum of the ratio $\frac{r}{c}$, which is the rate of return plus 1 (i.e. profit per dollar invested is maximised or minimised).

(E) $-\frac{5}{\sqrt{(x)}} = -\frac{10}{(x+1)^2}$

Let $a = \frac{r}{x}$ and $b = \frac{c}{x}$ be the average revenue per unit of production, and the average cost per unit of production, then this equation says $\frac{da}{dx} = \frac{db}{dx}$, i.e. the solution to this equation is the value that maximises (or minimises) $a - b = \frac{r-c}{x}$, that is, it maximises the profit per unit of production.

6. *If $z = x^2 + 3xy + 2xy^2 - 3y^3$, then $\frac{\partial z}{\partial y}$ at $x = 1$, $y = 2$ is*

$\frac{\partial z}{\partial y} = 0 + 3x + 2x(2y) - 9y^2$, so when $x = 1$ and $y = 2$, we have $\frac{\partial z}{\partial y} = 3 \times 1 + 2 \times 1 \times 2 \times 2 - 9 \times 2^2 = 3 + 8 - 36 = -25$,

(A) -25

(B) -24

(C) -9

(D) 1

(E) 16

7. *For two products, A and B, the demand functions for the products are given by:*

$$q_A = 1000 - \frac{p_A}{2000 - p_B} \tag{1}$$

$$q_B = \frac{300}{p_A + p_B} \tag{2}$$

(a) Calculate the partial derivatives $\frac{\partial q_A}{\partial p_A}$, $\frac{\partial q_A}{\partial p_B}$, $\frac{\partial q_B}{\partial p_A}$, and $\frac{\partial q_B}{\partial p_B}$. Determine whether the products are competitive or complementary (or neither).

$$\frac{\partial q_A}{\partial p_A} = -\frac{1}{2000 - p_B}$$

$$\frac{\partial q_A}{\partial p_B} = -\frac{p_A}{(2000 - p_B)^2}$$

$$\frac{\partial q_B}{\partial p_A} = -\frac{300}{(p_A + p_B)^2}$$

$$\frac{\partial q_B}{\partial p_B} = -\frac{300}{(p_A + p_B)^2}$$

It is easy to see that for $p_B < 2000$, $\frac{\partial q_A}{\partial p_B}$ and $\frac{\partial q_B}{\partial p_A}$ are both negative, so the products A and B are complementary.

(b) Suppose the products are produced by two different companies. Each company sets the price for their own product. What equations need to be solved so that neither company can increase their own revenue by changing price? [You do not need to solve the equations, but you must simplify them to equations involving only p_A and p_B. Hint: There are two equations that must be satisfied.]

If we let the revenues from products A and B be r_A and r_B respectively, then we have that $r_A = p_A q_A$ and $r_B = p_B q_B$. We want to find p_A and p_B such that r_A cannot be increased by changing p_A, and r_B cannot be increased by changing p_B. [This sort of situation is referred to as a Nash equilibrium.]

In terms of derivatives, this requires $\frac{\partial r_A}{\partial p_A} = 0$ and $\frac{\partial r_B}{\partial p_B} = 0$. Since $r_A = p_A q_A$, we have from the product rule that

$$\frac{\partial r_A}{\partial p_A} = p_A \frac{\partial q_A}{\partial p_A} + q_A = -\frac{p_A}{2000 - p_B} + 1000 - \frac{p_A}{2000 - p_B} = 1000 - 2\frac{p_A}{2000 - p_B}$$

and

$$\frac{\partial r_B}{\partial p_B} = p_B \frac{\partial q_B}{\partial p_B} + q_B = -\frac{300 p_B}{(p_A + p_B)^2} + \frac{300}{p_A + p_B} = \frac{300 p_A}{(p_A + p_B)^2}$$

so the equations that need to be solved are

$$\begin{aligned} 1000 - 2\frac{p_A}{2000 - p_B} &= 0 \\ \frac{300}{p_A + p_B} - \frac{300 p_B}{(p_A + p_B)^2} &= 0 \end{aligned}$$

[Alternatively, we can get rid of the fractions to get

$$\begin{aligned} p_A &= 500(2000 - p_B) \\ 300 p_A &= 0 \end{aligned}$$

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(c) Now suppose the products are both produced by the same company. Now what equations should be solved to find the prices p_A and p_B that the company should charge to maximise its total revenue. [Again, the equations should involve just p_A and p_B.]

This time, we want to maximise $r = r_A + r_B$. We calculated $\frac{\partial r_A}{\partial p_A}$ and $\frac{\partial r_B}{\partial p_B}$ in part (b). And we know that $\frac{\partial r_A}{\partial p_B} = p_A \frac{\partial q_A}{\partial p_B} = -\frac{{p_A}^2}{(2000 - p_B)^2}$, and $\frac{\partial r_B}{\partial p_A} = p_B \frac{\partial q_B}{\partial p_A} = -\frac{300 p_B}{(p_A + p_B)^2}$. We need to find the solution to $\frac{\partial r}{\partial p_A} = 0$ and $\frac{\partial r}{\partial p_B} = 0$.

That is, we need to find the solution to the equations:

$$\begin{aligned} 1000 - 2\frac{p_A}{2000 - p_B} - \frac{{p_A}^2}{(2000 - p_B)^2} &= 0 \\ \frac{300}{p_A + p_B} - \frac{300p_B}{(p_A + p_B)^2} - \frac{300p_B}{(p_A + p_B)^2} &= 0 \end{aligned}$$

[Again, we can clear the fractions and simplify to get

$$\begin{aligned} p_A(4000 + p_A - 2p_B) &= 1000(2000 - p_B)^2 \\ 300(p_A - p_B) &= (p_A + p_B)^2 \end{aligned}$$

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