MATH 2051, Problems in Geometry
Fall 2007
Toby Kenney

Midterm Examination

Wednesday 24th October, 10:35—11:20 AM

Friday 26th October, 10:35—11:20 AM

Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Section A – Wednesday 24th October, 10:35—11:20 AM

1 Let ABC be a triangle with incentre I, inradius r, and circumradius R. Let the feet of the perpendiculars from I to BC, AC and AB be D, E and F respectively.

(a) Show that $AF = s - a$ (where s is the semiperimeter and $a = BC$).

(b) By calculating FE in two different ways, show that $AI^2 = \frac{2r(s-a)}{\sin A}$.

(c) The same methods applied to DE and DF give $BI^2 = \frac{2r(s-a)}{\sin B}$ and $CI^2 = \frac{2r(s-c)}{\sin C}$. By cancelling various different expressions for the area (or otherwise) deduce that $AI.BI.CI = 4r^2R$.

Section B – Friday 26th October, 10:35—11:20 AM

2 Let ABC be a triangle such that all three angles are less than $120°$. Let P be a point in the triangle such that $\angle APB = \angle BPC = \angle CPA = 120°$. Let $x = AP$, $y = BP$, $z = CP$, $a = BC$, $b = AC$ and $c = AB$.

(a) Prove that $\triangle ABC = \frac{\sqrt{3}}{4}(xy + xz + yz)$.

(b) Prove that $2(x + y + z)^2 = (a^2 + b^2 + c^2) + 4\sqrt{3}\triangle ABC$.

[Hint: $\cos 120° = \frac{1}{2}$, $\sin 120° = \frac{\sqrt{3}}{2}$.]

3 Let $ABCD$ be a parallelogram, and let P, Q, R and S be internal points on AB, BC, CD and DA respectively (i.e. P lies between A and B etc.) such that $PQRS$ is a parallelogram. Let X be the point where PR and AC intersect. Prove that $AX = CX$.

4 Let $ABCD$ be a cyclic quadrilateral, with circumcircle Γ_1 having centre O_1. Let the diagonals AC and DB meet at X (inside Γ_1). Let Γ_2 and Γ_3 be the circumcircles of the triangles ABX and CDX respectively. Let Y be the other point where Γ_2 and Γ_3 meet (i.e. the point which is not X). Suppose Y is nearer than X to BC. Show that $OYBC$ is cyclic. [Hint: extend the line XY to a point Q past Y. Calculate $\angle BYC$ as $\angle BYQ + \angle QYC$.]