MATH 2051, Problems in Geometry
Fall 2007
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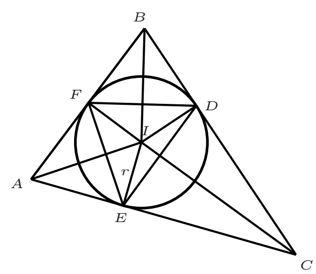
Midterm Examination
Wednesday 24th October, 10:35—11:20 AM
Friday 26th October, 10:35—11:20 AM
Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Section A

1 *Let ABC be a triangle with incentre I, inradius r, and circumradius R. Let the feet of the perpendiculars from I to BC, AC and AB be D, E and F respectively.*



(a) Show that $AF = s - a$ (where s is the semiperimeter and $a = BC$).

Since tangents from a point are equal, we know that $AF = AE$, $BF = BD$ and $CE = CD$, so $AF+AE = b+c-BF-CE = b+c-BD-CD = b+c-a$. Therefore, $AF = \frac{b+c-a}{2} = s - a$.

(b) By calculating FE in two different ways, show that $AI^2 = \frac{2r(s-a)}{\sin A}$, where $A = \angle BAC$.

Since $\angle IEA = \angle IFA = 90°$, the quadrilateral $AEIF$ is cyclic, and AI is a diameter, so by the extended sine rule on triangle AEF, $FE = AI \sin A$. On the other hand, if we let X be the point where AI meets FE, then from the right-angled triangles IFX and IEX, $FE = 2r \cos \frac{A}{2}$. From triangle AIE, we get that $\cos \frac{A}{2} = \frac{s-a}{AI}$. Therefore, $AI \sin A = FE = \frac{2r(s-a)}{AI}$, so $AI^2 = \frac{2r(s-a)}{\sin A}$

(c) The same methods applied to DE and DF give $BI^2 = \frac{2r(s-a)}{\sin B}$ and $CI^2 = \frac{2r(s-c)}{\sin C}$. By cancelling various different expressions for the area (or otherwise) deduce that $AI.BI.CI = 4r^2R$.

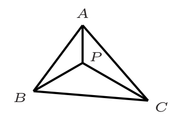
Multiplying these formulae together, we get:

$$AI^2BI^2CI^2 = \frac{8r^3(s-a)(s-b)(s-c)}{\sin A \sin B \sin C} = \frac{8R^2r^4s(s-a)(s-b)(s-c)}{srR^2 \sin A \sin B \sin C} =$$
$$\frac{16R^2r^4(\triangle ABC)^2}{(\triangle ABC)^2} = 16R^2r^4$$

so $AI.BI.CI = 4r^2R$.

Section B

2 *Let ABC be a triangle such that all three angles are less than* $120°$. *Let P be a point in the triangle such that* $\angle APB = \angle BPC = \angle CPA = 120°$. *Let* $x = AP$, $y = BP$, $z = CP$, $a = BC$, $b = AC$ *and* $c = AB$.



(a) Prove that $\triangle ABC = \frac{\sqrt{3}}{4}(xy + xz + yz)$.

$$\triangle ABC = \triangle BPC + \triangle APC + \triangle APB =$$
$$\frac{1}{2}yz \sin 120° + \frac{1}{2}xz \sin 120° + \frac{1}{2}xy \sin 120° =$$
$$\frac{\sqrt{3}}{4}(xy + xz + yz)$$

(b) Prove that $2(x + y + z)^2 = (a^2 + b^2 + c^2) + 4\sqrt{3}\triangle ABC$.

[Hint: $\cos 120° = \frac{1}{2}$, $\sin 120° = \frac{\sqrt{3}}{2}$.*]*

Using the cosine rule on triangle BPC, we get $a^2 = y^2 + z^2 + yz$. Using the cosine rule on the triangles APB and APC as well, and adding the three equations gives us

$$a^2 + b^2 + c^2 = 2(x^2 + y^2 + z^2) + xy + yz + xz$$

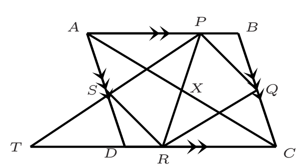
On the other hand,

$$2(x + y + z)^2 = 2(x^2 + y^2 + z^2) + 4(xy + yz + xz) =$$
$$a^2 + b^2 + c^2 + 3(xy + yz + xz)$$

From (a), we know that $\triangle ABC = \frac{\sqrt{3}}{4}(xy + xz + yz)$. Therefore, $3(xy + xz + yz) = 4\sqrt{3}\triangle ABC$, so $2(x + y + z)^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\triangle ABC$.

3 *Let ABCD be a parallelogram, and let P, Q, R and S be internal points on AB, BC, CD and DA respectively (i.e. P lies between A and B etc.) such that PQRS is a parallelogram.*

(a) Show that triangles APS and CRQ are congruent.

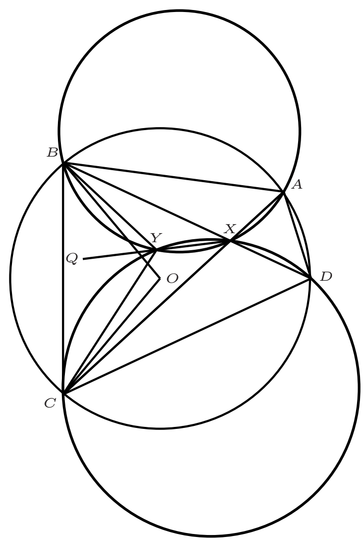


Extend PS and DC to meet at T. By alternate angles $\angle SPA = \angle STC$, and by corresponding angles, $\angle STC = \angle QRC$. Similarly, $\angle PSA = \angle RQC$. Also, since $PQRS$ is a parallellogram, $SP = RQ$, so triangles APS and CRQ are congruent by SAS.

(b) Let X be the point where PR and AC intersect. Prove that $AX = CX$.

Since triangles APS and CRQ are congruent, $AP = CR$. By alternate angles, $\angle ACR = \angle CAP$, and $\angle CRP = \angle APR$, so by ASA, triangles APX and CRX are congruent. Therefore, $AX = CX$.

4 *Let $ABCD$ be a cyclic quadrilateral, with circumcircle Γ_1 having centre O_1. Let the diagonals AC and DB meet at X (inside Γ_1). Let Γ_2 and Γ_3 be the circumcircles of the triangles ABX and CDX respectively. Let Y be the other point where Γ_2 and Γ_3 meet (i.e. the point which is not X). Suppose Y is nearer than X to BC. Show that $OYBC$ is cyclic. [Hint: extend the line XY to a point Q past Y. Calculate $\angle BYC$ as $\angle BYQ + \angle QYC$.]*



Since the angle at the centre is twice the angle at the circumference, $\angle BOC = 2\angle BAC$. On the other hand, since opposite angles in a cyclic quadrilateral add up to 180°, $\angle XYB = 180° - \angle BAC$, so $\angle QYB = \angle BAC$. Similarly, $\angle QYC = \angle BDC = \angle BAC$, so $\angle BYC = 2\angle BAC = \angle BOC$, so by the converse of angles in the same segment, $BOYC$ is cyclic.