MATH 2051, Problems in Geometry
Fall 2007
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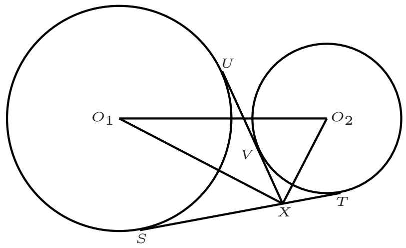
Mock Midterm Examination
Time allowed:
Section A – 45 minutes
Section B – 45 minutes
Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers.

Section A

1 *Let Γ_1 and Γ_2 be two disjoint circles with centres O_1 and O_2. Let l_1 be tangent to Γ_1 at S and tangent to Γ_2 at T, both on the same side of the line O_1O_2. Let l_2 be tangent to Γ_1 and Γ_2 at U and V respectively, on opposite sides of O_1O_2. Let l_1 and l_2 meet at X. Prove that $\angle O_1XO_2 = 90°$.*

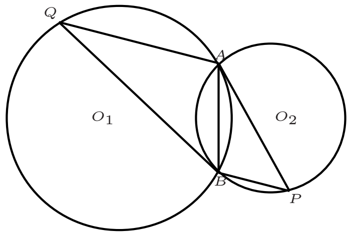


Since tangents from a point are equal, $XT = XV$, so by SSS, triangles XTO_2 and XVO_2 are congruent. Therefore, $\angle VXO_2 = \angle TXO_2$. Similarly, $\angle SXO_1 = \angle UXO_1$. Therefore

$$\angle O_1XO_2 = \angle O_1XU + \angle VXO_2 = \angle SXO_1 + \angle TXO_2 = 180° - \angle O_1XO_2$$

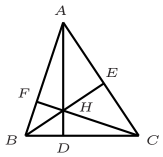
so $\angle O_1XO_2 = 90°$.

2 *Let Γ_1 and Γ_2 be two circles that intersect at A and B. Let the tangent to Γ_1 at A meet Γ_2 again at P, and let the tangent to Γ_2 at B meet Γ_1 again at Q. Prove that AQ and PB are parallel.*



By the alternate segment theorem $\angle QBA = \angle APB$ and $\angle PAB = \angle AQB$, so by angles in a triangle $\angle PBA = \angle QAB$, so by the converse of alternate angles, the lines QA and BP are parallel.

3 *Let ABC be a triangle, and let D, E and F be the feet of the altitudes from A, B and C respectively.*



(a) Find the lengths DE, DF and EF, and the angles $\angle DEF$, $\angle DFE$ and $\angle EDF$.

Note that the quadrilateral $BFEC$ is cyclic, so $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$, so triangles AEF and ABC are similar, so $\frac{EF}{BC} = \frac{AE}{AC} = \cos\angle BAC$. Therefore $EF = BC\cos\angle BAC$. Similarly, $DF = AC\cos\angle ABC$ and $DE = AB\cos\angle ACB$.

By angles in the same segment, $\angle DEH = \angle DCH = 90° - \angle ABC$ (from $\triangle BCF$. Similarly, $\angle FEH = 90° - \angle ABC$, so $\angle DEF = 180° - 2\angle ABC$, and similarly, $\angle EDF = 180° - 2\angle BAC$ $\angle DFE = 180° - 2\angle ACB$.

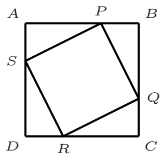
(b) Prove that the ratio of areas $\frac{\triangle DEF}{\triangle ABC} = 2\cos A\cos B\cos C$. [You may use the facts that $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin\theta = \sin(180° - \theta)$.]

The area of $\triangle DEF$ is

$$\frac{1}{2}DE.DF\sin\angle EDF = \frac{1}{2}AB.AC\cos C\cos B\sin(180° - 2A) =$$
$$\frac{1}{2}AB.AC\cos C\cos B\sin 2A = AB.AC\cos B\cos C\sin A\cos A =$$
$$2\triangle ABC\cos A\cos B\cos C$$

Section B

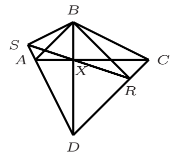
1 *Let $ABCD$ be a square of side 1, and let P, Q, R and S be points on AB, BC, CD, and DA respectively, such that $PQRS$ is a rectangle (all angles $90°$), and $AS < AP$ (so $\angle APS < 45°$). Prove that $PQRS$ is a square.*



Since $\angle SPQ = 90°$, $\angle QPB = 180° - \angle SPA - 90° = \angle PSA$. Also, $\angle PQB = \angle SPA$, so triangles PQB and SPA are similar. In the same way, triangles SPA, PQB, QRC, and RSD are all similar. Since $PQ = RS$, triangles PQB and RSD are congruent. Similarly, triangles SPA and QRC are congruent.

We therefore have that $\frac{AP}{BQ} = \frac{AS}{BP} = \frac{CQ}{BP} = \frac{1-BQ}{1-AP}$, so $AP(1-AP) = BQ(1-BQ)$. The solutions for this are $AP = BQ$ and $AP = 1-BQ = 1-DS = AS$. [We can get these by rearranging to $AP-BQ+BQ^2-AP^2 = 0$, and factorising as $(AP-BQ)(1-AP-BQ)$. It should be clear that they will both be solutions, so this algebra just shows that they are the only solutions.] In the first case, triangles ASP and BPQ are congruent, so $PQRS$ is a square. In the second case $\triangle ASP$ is isosceles which we know is not the case since $AP < AS$.

2 *Let $ABCD$ be a convex quadrilateral (all angles $< 180°$), such that the diagonals AC and BD meet at right angles at X. Let R and S be the feet of the perpendiculars from B to CD and DA respectively. Show that if $XS = XR$, then $AB.DC = BC.AD$.*



Note that $BCRX$ is cyclic (by the converse of angles in the same segment). Therefore, $\angle DRX = \angle DBC$, and $\angle DXR = \angle DCB$, so $\triangle DRX$ and $\triangle DBC$ are similar, so $\frac{RX}{BC} = \frac{DX}{DC}$. Similarly, $\frac{SX}{AB} = \frac{DX}{AC}$, so $\frac{AB}{BC} = \frac{RX.DX.AC}{SX.DX.DC}$. Cancelling and multiplying across, we get $AB.DC = BC.AD$.