

MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
Toby Kenney
Homework Sheet 10
Hints & Model Solutions

Compulsory questions

1 For each of the following relations, determine which of the four properties: reflexivity, symmetry, antisymmetry, and transitivity hold for that relation.

(a) The relation on \mathbb{N} that relates two natural numbers if their difference is at most 7.

This is reflexive and symmetric, since the difference between a number and itself is 0, which is at most 7, and the difference between m and n is the same as the difference between n and m ; but not transitive, since for example, 3 and 9 differ by at most 7, and 9 and 12 differ by at most 7, but 3 and 12 do not differ by at most 7.

(b) The relation “is an ancestor of” on the set of all people.

This is transitive, since an ancestor of an ancestor is still an ancestor. It is not symmetric, since people are not ancestors of their own ancestors. Indeed it is antisymmetric, since we never have a an ancestor of b and b an ancestor of a . I suppose reflexivity depends how you define the word ancestor. I think the normal usage would not make the relation reflexive – people are not generally considered ancestors of themselves.

(c) The relation “has the same birthday as” on the set of all people.

This is reflexive, symmetric and transitive, but not antisymmetric. Any person has the same birthday as themselves, if a has the same birthday as b , then b has the same birthday as a , and if a has the same birthday as b and b has the same birthday as c , then a must also have the same birthday as c . Finally, there are pairs of distinct people a and b who have the same birthday. (If we have time to cover the pigeon-hole principle, then this follows from that.) For such a pair, a is related to b , and b is related to a , but $a \neq b$.

(d) The relation “is a square root of” on the set of real numbers.

This is not reflexive, since for example, 2 is not a square root of 2. It is not symmetric, since 2 is a square root of 4 but 4 is not a square root of 2. It is antisymmetric, since if a is a square root of b and b is a square root of a , then we must have $a^4 = a$. The only solution to this is $a = 1$, and in that case $a^1 = 1$, so $b = 1$, and thus, $a = b$. It is not transitive, since 2 is a square root of 4 and 4 is a square root of 16, but 2 is not a square root of 16.

(e) The relation “is (strictly) taller than” on the set of all people.

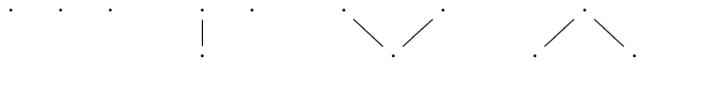
This is not reflexive or symmetric. It is transitive and antisymmetric, since if we never have that a is taller than b and b is taller than a , so the statement $(\forall a, b)((aRb \wedge bRa) \rightarrow a = b)$ is vacuously true.

(f) The relation on \mathbb{N} that relates two natural numbers if they are p^a and p^b for some prime p and positive integers a and b .

This is not reflexive, since for example, 6 is not related to itself, since it is not a power of a prime. (1 is not related to itself either). It is symmetric. It is also transitive, since by unique prime factorisation, a number n cannot be $p_1^{a_1}$ and $p_2^{a_2}$ for distinct primes p_1 and p_2 .

2 How many partial orders are there on a 3-element set (up to rearranging the elements of the set, so for example, $0j1j2$ and $1j0j2$ count as the same order)? Give all the corresponding Hasse diagrams.

There are 5 partial orders on a 3 element set. Here are the corresponding Hasse diagrams:



3 For each of the following functions, determine whether the function is injective, and whether it is surjective. Justify your answers.

(a) $f : \mathbb{R}^+ \rightarrow \mathbb{R}, f(x) = x^2$.

This is injective, since if $x^2 = y^2$ then $x^2 - y^2 = 0$, so $(x + y)(x - y) = 0$, so either $x = y$ or $x = -y$. However, $x = -y$ is impossible, since x and y must both be positive. Therefore, we must have $x = y$. It is not surjective, since for example, -1 is not the square of any positive real number.

(b) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$.

This is not injective, since for example, $f(2) = f(-2) = 4$, but $2 \neq -2$. It is not surjective since -1 is not the square of any real number.

$$(c) f : \{0, 1, 2, 3\} \rightarrow \{0, 1, 2\}, f(0) = 0, f(1) = 2, f(2) = 2, f(3) = 1.$$

This is surjective, since $f(0) = 0$, $f(3) = 1$ and $f(1) = 2$, so for each of $0, 1, 2$, there is an element of $\{0, 1, 2, 3\}$ that is sent to it by f . It is not injective, since $f(1) = f(2)$.

$$(d) f : \mathbb{Q} \rightarrow \mathbb{N}, f\left(\frac{a}{b}\right) = b \text{ whenever } (a, b) = 1.$$

This is not injective since for example, $f\left(\frac{1}{4}\right) = 4 = f\left(\frac{3}{4}\right)$. It is not surjective, since there is no rational number q with $f(q) = 0$.

$$(e) f : \mathbb{Z} \rightarrow \mathbb{N}, f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$

This is both injective and surjective: if n is negative, then $f(n)$ is odd, while if n is positive or 0, then $f(n)$ is even, so if $f(n) = f(m)$, then m and n are either both positive or 0, or both negative. In the first case, $2n = 2m$, so $m = n$. In the second, $-2n - 1 = -2m - 1$, so again $m = n$, so f is injective. On the other hand, if $k \in \mathbb{N}$ is even, then $k = 2l$ for some integer l , and l must be non-negative, since k is, so $k = f(l)$. If k is odd, then $k = 2l + 1$ for some integer l . Again, l must be non-negative since $k - 1$ is. Therefore, $k = f(-(l + 1))$. Therefore, f is surjective.

4 Suppose $f : B \rightarrow C$ and $g : A \rightarrow B$ are functions with composite $f \circ g : A \rightarrow C$. Give a proof or a counterexample for each of the following:

(a) If $f \circ g$ is injective then f is injective.

This is false, for example, if $A = \{0\}$, $B = \{0, 1\}$ and $C = \{0\}$, and f and g are given by $f(0) = f(1) = 0$, and $g(0) = 0$, then $f \circ g$ is injective, but f is not injective.

(b) If $f \circ g$ is injective then g is injective.

This is true.

Proof. Let a and b be any elements of A . Suppose that $g(a) = g(b)$. We need to show that $a = b$. Now, $f(g(a)) = f(g(b))$, so $f \circ g(a) = f \circ g(b)$. Since $f \circ g$ is injective, this means that $a = b$. Therefore, g must be injective. \square

(c) *If $f \circ g$ is surjective then f is surjective.*

This is true.

Proof. Let x be any element of C . We need to show that there is an element y of B such that $f(y) = x$. Since $f \circ g$ is surjective, we know that there is some element $z \in A$, such that $f \circ g(z) = x$. This means $f(g(z)) = x$. Therefore, if we let $y = g(z)$, then $y \in B$ and $f(y) = x$, so f is surjective. \square

(d) *If $f \circ g$ is surjective then g is surjective.*

This is false. Consider the same example as in (a). $f \circ g$ is surjective, since $f \circ g(0) = 0$. However, g is not surjective, since there is no $x \in \{0\}$ with $g(x) = 1$.