MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 2
Solutions and hints

Here are model solutions to the easier questions, and hints for the more difficult questions. For the questions where I have only given hints, you may submit revised solutions for half credit with your solutions to the next assignment. I will give out model solutions to those questions with the solutions and hints for that assignment.

These model solutions do not always list all possible ways of doing the questions.

Sheet 1

5 *Show that if for any propositions p, q, and r (not necessarily primitive propositions) $p \vee r \equiv p \vee q$ and $p \wedge r \equiv p \wedge q$ then we must have $q \equiv r$.*

$q \equiv q \vee (q \wedge p) \equiv q \vee (p \wedge q) \equiv q \vee (p \wedge r) \equiv (q \vee p) \wedge (q \vee r) \equiv (p \vee q) \wedge (q \vee r) \equiv (p \vee r) \wedge (q \vee r) \equiv (p \wedge q) \vee r \equiv (p \wedge r) \vee r \equiv r$

Alternatively, consider the truth table:

p	q	r	$p \vee q$	$p \vee r$	$(p \vee q) \leftrightarrow (p \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \leftrightarrow (p \wedge r)$
0	0	0	0	0	1	0	0	1
0	0	1	0	1	0	0	0	1
0	1	0	1	0	0	0	0	1
0	1	1	1	1	1	0	0	1
1	0	0	1	1	1	0	0	1
1	0	1	1	1	1	0	1	0
1	1	0	1	1	1	1	0	0
1	1	1	1	1	1	1	1	1

We see that $(p \vee q) \leftrightarrow (p \vee r)$ and $(p \wedge q) \leftrightarrow (p \wedge r)$ are both true if and only if q and r both have the same truth value.

6 *(c) From $p \to q$ and $(p \to r) \vee (q \to r)$, deduce $p \to r$.*

$p \to q$	
$(p \to r) \vee (q \to r)$	
$(q \to r) \to (q \to r)$	$p \to p$ is a tautology
$(p \to q) \vee \neg(q \to r)$	by generalisation from $p \to q$
$(q \to r) \to (p \to q)$	by logical equivalence
$(q \to r) \to (p \to r)$	by transitivity
$(p \to r) \to (p \to r)$	tautology
$p \to r$	by division into cases

Sheet 2

1 *Which of the following are true when* $A = \{0, 1, 2\}$ *and* $B = \{1, 2, 3, 4\}$*? Justify your answers.*

(a) $(\forall n \in A)(\exists m \in B)(m = n + 1)$

True – use the following values:

n	m
0	1
1	2
2	3

(b) $(\exists m \in B)(\forall n \in A)(m = n + 1)$

False – the following are counterexamples for each m:

m	n
1	1,2
2	0,2
3	0,1
4	0,1,2

(c) $(\exists n \in B)(n \in A)$

True – $n = 1$ or $n = 2$ are both examples.

(d) $(\exists m \in B)(\forall n \in A)(n < m)$

True – $m = 3$ and $m = 4$ both work.

(e) $(\forall m \in B)(m + 2 \in A \to m + 3 \in A)$

True – consider the truth table for each value of m:

m	$m+2 \in A$	$m+3 \in A$	$(m+2 \in A) \to (m+3 \in A)$
1	$\perp$	$\perp$	$\top$
2	$\perp$	$\perp$	$\top$
3	$\perp$	$\perp$	$\top$
4	$\perp$	$\perp$	$\top$

(f) $(\exists m \in B)(m+2 \in A \wedge m+3 \in A)$

False – Now the truth table is:

m	$m+2 \in A$	$m+3 \in A$	$(m+2 \in A) \wedge (m+3 \in A)$
1	$\perp$	$\perp$	$\perp$
2	$\perp$	$\perp$	$\perp$
3	$\perp$	$\perp$	$\perp$
4	$\perp$	$\perp$	$\perp$

2 Write down the negations of the following statements. (The answers given here are possible answers – any one of them is an acceptable answer, and there may be other possible answers.)

(a) All students in this course will get "A"s.

At least one student in this course will not get an "A".

Some students in this course will not get "A"s.

(b) Every student will miss at least one lecture. (i.e. For every student, there will be at least one lecture that student misses.)

At least one student will attend every lecture.

Some students will attend all the lectures.

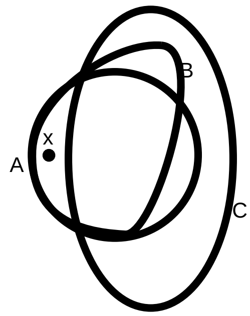
(c) There will be a lecture that every student attends.

In every lecture, at least one student will not attend.

3 Use Venn diagrams to show the following arguments are not valid:

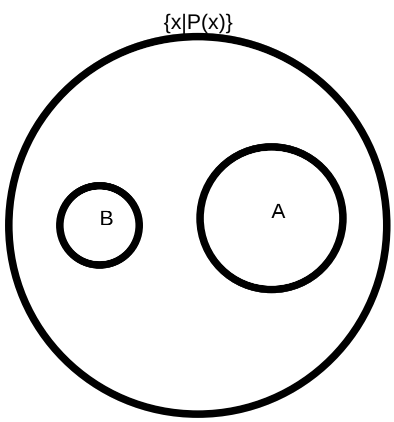
(a)

$$(\forall x \in A)(x \in B \vee x \in C)$$
$$(\forall x \in B)(x \in A \vee x \in C)$$
$$\therefore (\forall x \in B)(x \in C)$$



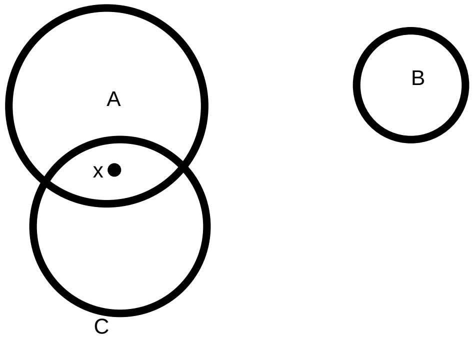
(b)

$$(\forall x \in A)(P(x))$$
$$(\forall x \in B)(\neg x \in A)$$
$$\therefore (\exists x \in B)(\neg P(x))$$



(c)

$$(\forall x \in A)(\neg x \in B)$$
$$(\forall x \in B)(\neg x \in C)$$
$$\therefore (\forall x \in A)(\neg x \in C)$$



4 Show that the following arguments are valid using universal instantiation and the rules of inference in Chapter 1:

(a)

$$\phi(x)$$
$$\therefore (\exists y)(\phi(y))$$

Hint:

The statement $(\exists y)(\phi(y))$ is equivalent to $\neg(\forall y)(\neg\phi(y))$, which is equivalent to $((\forall y)(\neg\phi(y))) \to 0$. You can get a proof of this from $\phi(x)$ from a proof of 0 from $(\forall y)(\neg\phi(y))$ and $\phi(x)$ by starting with:

$\phi(x)$	
$\phi(x) \vee \neg(\forall y)(\neg\phi(y))$	by generalisation
$(\forall y)(\neg\phi(y)) \to \phi(x)$	by $p \to q \equiv q \vee \neg p$
$(\forall y)(\neg\phi(y)) \to (\forall y)(\neg\phi(y))$	tautology

and putting all the steps on the right of $(\forall y)(\neg\phi(y)) \to$.

(b)

$$(\forall x \in A)(x \in B)$$
$$(\exists x \in A)(x \in C)$$
$$\therefore (\exists y \in B)(y \in C)$$

I think this question is in fact not possible using universal instantiation – I think you need to use generalisation (covered in Chapter 3) to do it. Sorry about that – I should check my questions more carefully.

(c)

$$
\begin{aligned}
&(\forall x \in A)(x \in B \to x \in C)\\
&(y \in A)\\
&y \in C \to \neg(y \in B)\\
&\therefore \neg y \in B
\end{aligned}
$$

$(\forall x \in A)(x \in B \to x \in C)$	
$y \in A$	
$(y \in B) \to (y \in C)$	by instantiation
$(y \in C) \to \neg(y \in B)$	premise
$(y \in B) \to \neg(y \in B)$	by transitivity
$(y \in B) \to (y \in B)$	tautology
$(y \in B) \to 0$	since $(y \in B) \wedge \neg(y \in B)$ is a contradiction
$\neg(y \in B)$	

(d)

$$
\begin{aligned}
&(\forall x \in A)(x \in B \vee x \in C)\\
&y \in A \wedge \neg(y \in B)\\
&\therefore (y \in C)
\end{aligned}
$$

$(\forall x \in A)(x \in B \vee x \in C)$	
$y \in A \wedge \neg(y \in B)$	
$y \in A$	by specialisation
$(y \in B) \vee (y \in C)$	by instantiation
$\neg(y \in B)$	by specialisation
$y \in B$	by elimination

5 *Another quantifier that is sometimes used is $(\exists! x)(\phi(x))$, meaning that there is exactly one x such that $\phi(x)$ is true. Rewrite this expression using $\exists$ and $\forall$.*

Hint:

This statement is a conjunction of the statements "There is at least one x such that $\phi(x)$ holds.", and "There is at most one x such that $\phi(x)$ holds." To express the latter, you'll need to use the equality predicate.