MATH 2112/CSCI 2112, Discrete Structures I
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Homework Sheet 4
Hints & Model Solutions

Sheet 3

2 *e) Prove or disprove that there is a positive integer n such that n, $n+1$, $n+2$, ..., $n+100$ are all composite.*

This is true, let $n = 2\times 3\times \ldots \times 102+2$. Then $n = 2\times(3\times\cdots\times 102+1)$, and for any integer $0 \leqslant k \leqslant 100$, $n+k = (k+2)\times(2\times 3\times\cdots\times(k+1)\times(k+3)\times\cdots\times 102+1)$, so it is not prime.

Sheet 4

1 *Show that for all integers n, n^2 is congruent to 0, 1, 2, or 4 modulo 7.*

n is congruent to one of 0, 1, 2, 3, 4, 5, or 6 modulo 7 (by the Quotient Remainder Theorem). If $n \equiv 0$ then $n^2 \equiv 0$; if $n \equiv 1$ or $n \equiv 6$, then $n^2 \equiv 1$; if $n \equiv 2$ or $n \equiv 5$, then $n^2 \equiv 4$; finally if $n \equiv 3$ or $n \equiv 4$, then $n^2 \equiv 2$, so in all cases, n^2 is congruent to 0, 1, 2, or 4 modulo 7.

2 *Show that $2^{13} + 3^{241}$ is divisible by 5.*

$2^4 = 16 \equiv 1 \pmod 5$, so $2^13 = 2\times(2^4)^3 \equiv 2 \pmod 5$. Also, $3^4 = 81 \equiv 1 \pmod 5$, so $3^{241} = 3\times(3^4)^{60} \equiv 3 \pmod 5$. Therefore, $2^{13}+3^{241} \equiv 0 \pmod 5$, so it is divisible by 5.

3 *If $7|n^3$ for some integer n, show that $7|n$.*

n^3 can be expressed uniquely as a product of prime numbers. Since $7|n^3$, one of these prime numbers must be 7. Now consider the prime factorisation $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$. $n^3 = p_1^{3\alpha_1}p_2^{3\alpha_2}\ldots p_k^{3\alpha_k}$ is the unique prime factorisation of n^3, so one of $p_1, p_2, \ldots, p_k$ must be 7. Therefore, $7|n$.

4 *Show that for any integer n, one of n^2+4 and n^3+n is divisible by 5.*

Any integer is congruent to one of 0, 1, 2, 3, or 4 modulo 5. If it is congruent to 1 or 4 mod 5, then $n^2+4 \equiv 0$, while if it is 0, 2, or 3, then $n^3+n \equiv 0$, so in either case one of them is divisible by 5.

5 *Define the repeat of a positive integer as the number obtained by writing it twice in a row (in decimal). For example, the repeat of 364 is 364364. Find a positive integer n such that the repeat of n is equal to* m^2 *for some integer m. [Hint: the repeat of n is always a multiple of n. You may find the following divisibility test useful: a number is divisible by 11 if and only if the sum of its odd digits minus the sum of its even digits is divisible by 11. For example, 1254 is divisible by 11 since the sum of its odd digits is 1+5=6 and the sum of its even digits is 2+4=6, so their difference is 0, which is divisible by 11. You won't find the answer by trial and error.]*

Hint:

The repeat of a number n with k digits is $(10^k + 1)n$. If this is a square, then every prime factor of $10^k + 1$ must divide it, and must therefore also divide its square root. Thus, the square of every prime factor of $10^k + 1$ must divide $(10^k + 1)n$, so either its square will divide $10^k + 1$, or it must divide n. Since $n < 10^k + 1$, if $10^k + 1$ is not divisible by the square of any prime number, then $(10^k + 1)n$ cannot be a square. Therefore, we have to look for a positive integer k such that $10^k + 1$ is divisible by the square of some prime. Try looking for some k such that $10^k + 1$ is divisible by the square of 11.

6 *Show that* $3^{258} + 17$ *is not the square of any integer n.*

$3^{258} + 17 \equiv 17 \equiv 2 \pmod 3$. However, if $n \equiv 0 \pmod 3$, then $n^2 \equiv 0 \pmod 3$, if $n \equiv 1 \pmod 3$ then $n^2 \equiv 1 \pmod 3$, and if $n \equiv 2 \pmod 3$, then $n^2 \equiv 1 \pmod 3$, so n^2 is never congruent to 2 modulo 3. Therefore, $3^{258} + 17$ cannot be the square of an integer.

7 *Solve for x – i.e. find all values of x less than the number modulo which we are working (so, for example, in (a), give all values of* $x \in \{0, 1, \ldots, 12\}$*) satisfying the equation:*

(a) $4x \equiv 7 \pmod{13}$

$3 \times 4 = 12 \equiv -1 \pmod{13}$, while $7 \equiv -6 \pmod{13}$, so $7 \equiv 6 \times (3 \times 4) \pmod{13}$ $6 \times 3 = 18 \equiv 5 \pmod{13}$, so $4 \times 5 \equiv 7 \pmod{13}$. 4 and 13 have no common factor greater than 1, so if $4x \equiv 4y \pmod{13}$, then $13|4(x - y)$, and therefore, by unique prime factorisation, $13|(x - y)$, so $x \equiv y \pmod{13}$. Thus $x \equiv 5 \pmod{13}$ is the unique solution.

(b) $8x \equiv 9 \pmod{12}$

$8x$ is always divisible by 4, but $12m + 9 = 4(3m + 2) + 1$ is never divisible by 4, so no number that is congruent to 9 modulo 12 can be divisible by

8, so there are no solutions.

(c) $7x \equiv 19 \pmod{24}$

$7 \times 7 \equiv 1 \pmod{24}$, so $7 \times (7 \times 19) \equiv 19 \pmod{24}$. $7 \times 19 = 133 \equiv 13 \pmod{24}$, so $x \equiv 13 \pmod{24}$ is a solution. 7 and 24 have no non-trivial common factors, so as in (a), $x \equiv 13 \pmod{24}$ is the only solution.

(d) $12x \equiv 44 \pmod{64}$

This equation says: $12x - 44 = 64k$ for some integer k. Dividing through by 4, we get $3x - 11 = 16k$, i.e. $3x \equiv 11 \pmod{16}$. $3 \times 5 \equiv -1 \pmod{16}$, so $3 \times 5 \times 5 \equiv -5 \equiv 11 \pmod{16}$. $25 \equiv 9 \pmod{16}$. Therefore, $x \equiv 9 \pmod{16}$ is the only solution to $3x \equiv 11 \pmod{16}$. This gives us the solutions $x \equiv 9 \pmod{64}$, $x \equiv 25 \pmod{64}$, $x \equiv 41 \pmod{64}$ and $x \equiv 57 \pmod{64}$ as the solutions to the original equation.

8 *Consider the integers whose last digit (in decimal) is 1. The product of any two such integers is another such integer. Any such integer can therefore be factored as a product of integers of this type that cannot be written as non-trivial products of other integers of this type. For example,* $7,211 = 11 \times 21 \times 31$*, and 11, 21, and 31 cannot be expressed as products of integers whose last digit is 1.*

Can every integer of this type be written in a unique way as such a product? Give a proof or a counterexample.

Hint:

Try to find prime numbers p_1, p_2, p_3, p_4 such that none of them has last digit 1 in decimal, but the products p_1p_2, p_1p_3, p_3p_4, and p_2p_4 all have last digit 1. Since none of p_1, p_2, p_3, and p_4 has last digit 1, the products p_1p_2, p_1p_3, p_3p_4, and p_2p_4 cannot be expressed as products of smaller numbers with last digit 1, so $p_1p_2 \times p_3p_4$ and $p_1p_3 \times p_2p_4$ will be two factorisations of $p_1p_2p_3p_4$.

9 *Are there any solutions to* $x^6 + y^6 + 3 = z^6$*, where* x*,* y*, and* z *are integers? [Hint: any solution* x, y, z *would also be a solution to* $x^6 + y^6 + 3 \equiv z^6 \pmod{n}$ *for any integer* n*.]*

A solution to this equation would also be a solution to the congruence $x^6 + y^6 + 3 \equiv z^6 \pmod{7}$. However, modulo 7 $0^6 \equiv 0$, $1^6 \equiv 1$, $2^6 \equiv 1$, $3^6 \equiv 1$, $4^6 \equiv 1$, $5^6 \equiv 1$, and $6^6 \equiv 1$. Therefore, the possible values of $x^6 + y^6 + 3$ modulo 7 are 3, 4, and 5, while the possible values of z^6 are

0 and 1. Therefore, this congruence has no solutions, and so the equation has no solutions in integers.