MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 6
Due in: Monday 12th March, 1:30 PM

Compulsory questions

1 Show that if $m > 1$ and $n > 1$ are natural numbers such that $6|mn$, then it is possible to cover an $m \times n$ chessboard with 3×2 tiles. [Hint: if $3|m$ and $2|n$, or $2|m$ and $3|n$, this should be easy. If $6|m$ and $n > 2$, divide into two cases: $n = 2k + 3$ and $n = 2k$. Prove each of these by induction on k.]

2 Consider the set of ordered pairs (m, n) of natural numbers, ordered by $(k, l) < (m, n)$ if either $k < m$ or ($k = m$ and $l < n$). [This is called the lexicographic order; it is the way words are ordered in the dictionary.] For example, $(1, 7) < (2, 1)$, and $(3, 4) < (3, 5)$. Show that this set is a well-order.

3 Show that $\sum_{i=1}^{n} i^2(i + 1) = \frac{n(n+1)(n+2)(3n+1)}{12}$.

4 What is wrong with the following "proof" that all maths lecturers are the same age?

Claim. *All maths lecturers are the same age.*

Proof. By induction on the number of maths lecturers. If there is only one maths lecturer, the claim is obvious. Now suppose the claim is true for any set of at most n maths lecturers. We want to prove that it is true for any set of at most $n + 1$ maths lecturers. Let $l_1, \ldots, l_{k+1}$ be a set of $k + 1$ maths lecturers. By our induction hypothesis, all lecturers in the set $l_1, \ldots, l_k$ are the same age, and also, all lecturers in the set $l_2, \ldots, l_k + 1$ are the same age. Let a_1 be the age of all of $l_1, \ldots, l_k$, and let a_2 be the age of $l_2, \ldots, l_{k+1}$. But the lecturers $l_2, \ldots, l_k$ are in both sets, so their ages must be both a_1 and a_2. Therefore, a_1 and a_2 must be equal. Thus, all of $l_1, \ldots, l_{k+1}$ are the same age.

Therefore, by induction, all maths lecturers are the same age. □

5 Prove that if $m, n < 2^k$ then Euclid's algorithm finds the greastest common divisor of m and n in at most $2k$ steps. [Hint: how large are the numbers r_0 and r_1?]

6 In Sheet 4, Question 3 (a), you were asked to prove that any positive integer congruent to 3 modulo 4 is divisible by a prime that is also congruent to 3 modulo 4. You did this by contradiction, using the fact that the

product of any collection of primes all congruent to 1 modulo 4 is also congruent to 1 modulo 4 (proving this requires induction). Now prove the same result by strong induction. [Hint: If n is prime, the result is obviously true. If not, then $n = ab$, where a and b must both be odd, $a > 1$ and $b > 1$, and one of them must be congruent to 3 modulo 4.]

Bonus Question

7 An $n \times n$ magic square is an $n \times n$ array containing each of the numbers $1, \ldots, n^2$ exactly once, such that every row, column and diagonal has the same sum. The following is a 3×3 magic square:

2	9	4
7	5	3
6	1	8

Show that for any positive integer, k, there is a $3^k \times 3^k$ magic square.