MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Make-up Midterm Examination
Wednesday 28th February: 6:00-7:30 PM

Answer all questions.

1 Which of the following are true when $A = \{1, 3, 7\}$ and $B = \{0, 4, 6, 10, 12, 34\}$? Justify your answers.

(a) $(\exists x \in A)(\forall y \in B)(x + y \text{ is prime})$

(b) $(\forall x \in A)(\exists y \in B)(x + y \text{ is prime})$

2 Use Euclid's algorithm to find the greatest common divisor of 193 and 114. Write down all the steps involved. Use your calculations to find integers a and b such that $193a + 114b$ is the greatest common divisor of 193 and 114.

3 Use universal instantiation and rules of inference to show that the following argument is valid.

$$(\forall x \in A)(\neg(x \in B))$$
$$(y \in A \vee y \in C) \wedge (y \in B \vee y \in C)$$
$$\therefore y \in C$$

4 Which of the following pairs of propositions are logically equivalent? Justify your answers.

(a) $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$.

(b) $p \vee (\neg q \to r)$ and $q \vee (\neg p \to r)$.

5 Use a Venn diagram to show the following argument is invalid:

$$(\forall x \in A)(x \in B)$$
$$(\exists x \in B)(x \in C)$$
$$\therefore (\exists x \in A)(x \in C)$$

6 Prove or disprove the following. You may use results proved in the course or the homework sheets, provided you state them clearly.

(a) There are infinitely many primes congruent to either 2 or 3 modulo 5. [You may assume that any integer that is congruent to 2 or 3 modulo 5 is divisible by a prime number congruent to 2 or 3 modulo 5. You may also assume that if n is not divisible by 5, then $n^4 \equiv 1 \pmod 5$. Hint: consider $(p_1 p_2 \cdots p_k)^4 + 1$.]

(b) $\sqrt[3]{16}$ is irrational.

(c) There is a natural number n such that $2n^2 + 3n + 1$ is prime.

(d) There is a natural number n such that $n^2 + 4n - 6$ is prime.

(e) $2^{12} + 3^{26} + 5^{29}$ is divisible by 11.

(f) For all natural numbers n, $\frac{n^3+5n+6}{3} = 2^{n+1}$.

7 Find an integer k, such that for all natural numbers n, $\sum_{i=1}^{n} \frac{i(i+1)(2i+1)}{6} = \frac{n(n+1)^2(n+2)}{k}$. Prove that the formula works for your value of k. [Hint: try to prove the result by induction. The proof will only work for one value of k.]

8 Find $0 \leqslant n < 840$ satisfying all the following congruences:

$$n \equiv 5 \pmod 8 \quad (1)$$
$$n \equiv 4 \pmod{15} \quad (2)$$
$$n \equiv 6 \pmod 7 \quad (3)$$

9 Find a boolean expression for the following logic circuit.