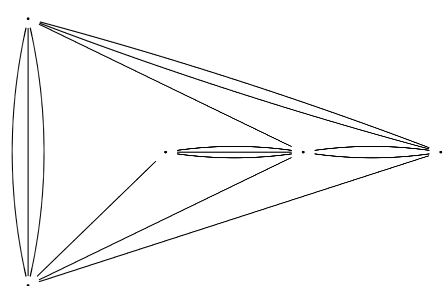
MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
Toby Kenney
Final Examination
Saturday 19th April, 14:00—17:00

Calculators **not** permitted. Justify all your answers.

Compulsory questions

1 (a) Write down the adjacency matrix for the graph:



(b) The cube of the adjacency matrix is

$$A^3 = \begin{pmatrix} 26 & 24 & 60 & 45 & 45 \\ 24 & 6 & 36 & 53 & 17 \\ 60 & 36 & 28 & 40 & 34 \\ 45 & 53 & 40 & 24 & 52 \\ 45 & 17 & 34 & 52 & 24 \end{pmatrix}$$

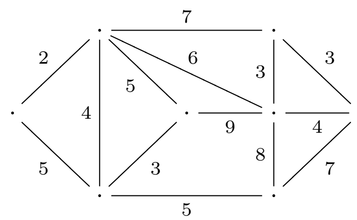
(This may be different from the cube of the matrix you wrote down in (a) because the rows and columns may be in a different order.) How many triangles are there in the graph? [Order doesn't matter, so *abc* and *bac* count as the same triangle.]

2 I have a standard deck of cards – there are 52 cards: 13 of each suit and 4 of each rank.

(a) If I pick one card from the deck. Are the events "The card is an Ace." and "The card is a spade." independent? Justify your answer fully.

(b) If I pick two cards from the deck without replacement. Are the events "Both of the cards are aces." and "Neither of the cards is a spade." independent? Justify your answer fully.

3 Find a minimum spanning tree for the following graph:



4 I have 6 dice, all fair dice with sides labelled 1, 2, 3, 5, 7, and 11. I roll all 6 and take the product.

(a) How many different products are possible? [Hint: by unique prime factorisation, the only way to get the same product is to roll the same multiset of numbers, possibly in a different order.]

(b) What is the most likely product?

5 Show that if we have 5 triangles in a K_6, then some 2 share an edge. [Hint: What is the largest number of triangles that can share a given vertex with no two sharing an edge?]

6 n fair dice are rolled. What is the probability that the highest number rolled is a 4?

7 We 2-colour the edges of a K_{17}. Show that either there is a monochromatic K_4, or the red edges form a graph with an Euler circuit. [You may assume that $R(3,4) = 9$.] (2 marks)

8 Let p_n be the probability of tossing a fair coin n times without getting 2 consecutive heads. Show that $p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2}$.

9 (a) Draw two graphs with the same degree sequence which are not isomorphic.

(b) Draw two connected simple graphs with the same degree sequence which are not isomorphic.

10 Let X be a random variable with values from $\{1, 2, \ldots, 100\}$ each occuring with probability $\mathbb{P}(X = n) = \frac{1}{100}$. Let Y be the total obtained by rolling 16 independant fair dice and taking the total. What is the probability that $Y \geqslant X$?

[Hint: how is this probability related to $\mathbb{E}(Y)$? Work out the probability conditional on a fixed value of Y first. You may assume that the expectation of a single roll of a fair die is $\frac{7}{2}$.]

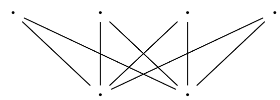
11 One person in 100 has a certain disease (i.e.) a randomly chosen person has a 1% (0.01) chance of having the disease, and a 99% (0.99) chance of not having it. There is a test, which when given to a person with the disease has a 90% (0.9) probability of giving a positive result. When given to a person without the disease, it has a 2% (0.02) probability of giving a positive result. A person is chosen at random and tested.

(a) What is the probability that the test gives a positive result?

(b) Given that the test gives a positive result, what is the probability that the person actually has the disease?

12 2-colour the edges of a K_7 red with probability $\frac{1}{2}$ and blue with probability $\frac{1}{2}$.

(a) What is the expected number of monochromatic $K_{4,2}$ (complete bipartite graphs on sets of 4 and 2 vertices) – e.g.



(b) Deduce that it is possible to 2-colour a K_7 without a monochromatic $K_{2,4}$.