MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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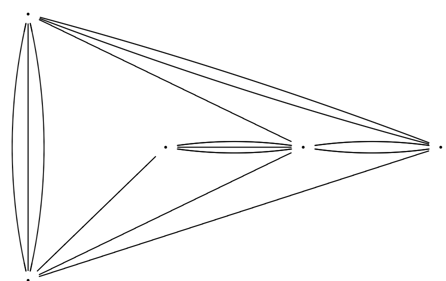
Final Examination
Hints & Model Solutions

Justify all your answers.

Compulsory questions

1 (a) Write down the adjacency matrix for the graph:



The adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3 & 0 \\ 3 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 & 0 \end{pmatrix}$$

(You can permute rows and columns if you put the vertices in a different order.)

(b) The cube of the adjacency matrix is

$$A^3 = \begin{pmatrix} 26 & 24 & 60 & 45 & 45 \\ 24 & 6 & 36 & 53 & 17 \\ 60 & 36 & 28 & 40 & 34 \\ 45 & 53 & 40 & 24 & 52 \\ 45 & 17 & 34 & 52 & 24 \end{pmatrix}$$

How many triangles are there? [Order doesn't matter, so abc and bac count as the same triangle.]

We add up the diagonal entries to get the number of ordered closed walks of length 3 is 108. Every closed walk defines a triangle, because there are no loops, so a vertex cannot be repeated. However each triangle gives rise to 6 different closed walks, so there are 18 triangles in total.

2 *I have a standard deck of cards – there are 52 cards: 13 of each suit and 4 of each rank.*

(a) If I pick one card from the deck. Are the events "The card is an Ace." and "The card is a spade." independent?

The probability that the card is an ace is $\frac{4}{52} = \frac{1}{13}$. The probability that it is a spade is $\frac{13}{52} = \frac{1}{4}$. The probability that it is an ace and a spade is $\frac{1}{52} = \frac{1}{13} \times \frac{1}{4}$, so they are independant.

(b) If I pick two cards from the deck without replacement. Are the events "Both the cards are aces." and "Neither of the cards is a spade." independent?

The probability that both cards are aces is $\frac{\binom{4}{2}}{\binom{52}{2}}$. The probability that neither is a spade is $\frac{\binom{39}{2}}{\binom{52}{2}}$. (For there to be no spade, both cards must be chosen from the remaining 39 cards.) The probability that there is are no spades and both cards are aces is $\frac{\binom{3}{2}}{\binom{52}{2}}$. (We now have 3 aces from which to choose 2.)

For the events to be independant, we need that

$$\frac{\binom{3}{2}}{\binom{52}{2}} = \frac{\binom{39}{2}}{\binom{52}{2}} \times \frac{\binom{4}{2}}{\binom{52}{2}}$$

or equivalently that

$$\binom{3}{2}\binom{52}{2} = \binom{39}{2}\binom{4}{2}$$

Since $\binom{4}{2} = 6$ and $\binom{3}{2} = 3$, we need $\frac{\binom{52}{2}}{\binom{39}{2}} = 2$ or

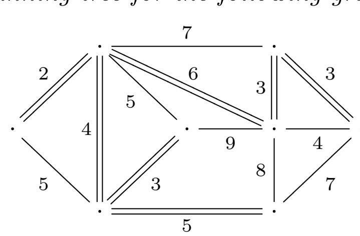
$$\frac{52 \times 51}{39 \times 38} = 2$$

This is not the case. In fact

$$\frac{52 \times 51}{39 \times 38} = \frac{4 \times 17}{1 \times 38} = \frac{2 \times 17}{19}$$

So the events are not independent.

3 *Find a minimum spanning tree for the following graph:*



4 *I have 6 dice, all fair dice with sides labelled 1, 2, 3, 5, 7, and 11. I roll all 6 and take the product.*

(a) How many different products are possible? [Hint: by unique prime factorisation, the only way to get the same product is to roll the same numbers, possibly in a different order.]

By unique prime factorisation, for two rolls to give the same product, they must be the same numbers in a different order. Therefore, this question is equivalent to the question "How many multisets of 6 elements can we form from the set $\{1, 2, 3, 5, 7, 11\}$?" We can determine this by considering x_1 is the number of ones, x_2 is the number of 2s, and so on. Now if we let $y_0 = x_1$, $y_1 = x_1 + x_2 + 1$, $y_2 = x_1 + x_2 + x_3 + 2$, $y_3 = x_1 + x_2 + x_3 + x_5 + 3$, and $y_4 = x_1 + x_2 + x_3 + x_5 + x_7 + 4$, then $y_0, \ldots, y_4$ are a strictly increasing sequence of numbers from $\{0, 1, \ldots, 6 + 4\}$, and from any such sequence, we can reconstruct $x_1, x_2, x_3, x_5, x_7, x_{11}$. The number of such sequences is $\binom{11}{5}$ (=462).

Alternatively, draw x_1 crosses then a vertical line, then x_2 crosses, then a vertical line, and so on until we draw x_{11} crosses but no vertical line. We have drawn a total of 11 symbols, 5 of which are vertical lines. Given any arrangement of such symbols, we can find exactly one arrangement of $x_1, x_2, x_3, x_5, x_7, x_{11}$ to generate this arrangement. For example, $xx|xxx||||x$ corresponds to $x_1 = 2, x_2 = 3, x_3 = x_4 = x_5 = 0, x_{11} = 1$, or to the roll $1, 1, 2, 2, 2, 11$, and so to the product $1 \times 1 \times 2 \times 2 \times 2 \times 11 = 88$. The number of arrangements of such symbols is $\binom{11}{5}$ (we just choose the positions of the vertical lines).

(b) What is the most likely product?

The number of ways to get a particular product is just the number of different ways to get a particular sequence of rolls of the dice. The largest this can be is 6!, because this is the number of ways we can redistribute the rolls among the dice. However, if any two rolls are the same, then swapping those two dice is still the same roll, so there are fewer ways to get that roll, and therefore fewer ways to get that product. The most likely product is therefore the one formed when each die shows a different number, i.e. $1 \times 2 \times 3 \times 5 \times 7 \times 11$. (=2320).

5 *Show that if we have 5 triangles in a K_6, then some 2 share an edge. – i.e. it is not possible to express K_6 as a union of edge-disjoint triangles. [Hint: What is the largest number of edge-disjoint triangles that can share a given vertex?]*

Each vertex is incident with 5 edges. Each vertex of a triangle is incident with 2 of the edges of the triangle. Therefore, if we have 3 triangles sharing a vertex, they give a total of 6 edges incident with that vertex. By the pigeon-hole principle, some two of these edges must be the same.

If we have 5 triangles, each of them has 3 vertices, for a total of 15 vertices. Therefore, by the generalised pigeon-hole principle, there must be some vertex shared by 3 triangles. Therefore, by the above argument, some 2 triangles share an edge.

6 *n fair dice are rolled. What is the probability that the highest number rolled is a 4?*

The probability that all the numbers are 4 or less is $\left(\frac{4}{6}\right)^n$. The probability that all the numbers are 3 or less is $\left(\frac{3}{6}\right)^n$. The highest number is a 4 if and only if all the numbers are 4 or less, but not all the numbers are 3 or less. The probability of this is $\frac{4^n-3^n}{6^n}$.

7 *We 2-colour the edges of a K_{17}. Show that there is either a monochromatic K_4, or the red edges form a graph with an Euler circuit. [You may assume that $R(3,4)=9$.]*

Take a vertex v_0. It has 16 neighbours. If 9 of the edges incident with v_0 are red, then let the neighbours along these edges be $v_1, v_2, \ldots, v_9$. Since $R(3,4)=9$, there is either a red K_3 or a blue K_4 among $v_1, \ldots, v_9$. In the second case, there is a blue K_4. In the first case, v_0 together with these vertices form a red K_4, so in either case, there is a monochromatic K_4. A similar argument shows that if any vertex is incident with 9 blue edges, then there is a monochromatic K_4.

Since every vertex has 16 neighbours, there are 3 possibilities:

i There is a vertex incident with 9 red edges.

ii There is a vertex incident with 9 blue edges.

iii Every vertex is incident with exactly 8 red edges and 8 blue edges.

We have shown that in cases (i) and (ii), we get a monochromatic K_4. In case (iii), in the graph formed by just the red edges, every vertex has degree 8, which is even, so the graph formed by just the red edges has an Euler circuit.

8 *Let p_n be the probability of tossing a fair die n times without getting 2 consecutive heads. Show that $p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2}$.*

We will divide into 2 events:

i The first toss is a tail. (And we never have two consecutive heads in the remaining $n-1$ tosses.)

ii The first toss is a head. Therefore, the second toss is a tail, and we never have two consecutive heads in the remaining $n-2$ tosses.

These cases are clearly disjoint, and clearly cover the whole event. p_n is therefore the sum of the probabilities of each of these events. The first event has probability $\frac{1}{2}p_{n-1}$ (The probability that the first roll is a tail, and that in the $n-1$ subsequent rolls, there are not 2 consecutive heads). The second event has probability $\frac{1}{4}p_{n-2}$ so we get

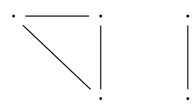
$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2}$$

9 *(a) Draw two graphs with the same degree sequence which are not isomorphic.*

The simplest example is

and

Another example using simple graphs is



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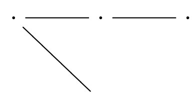
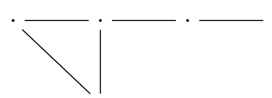


(b) Draw two connected simple graphs with the same degree sequence which are not isomorphic.

One example is



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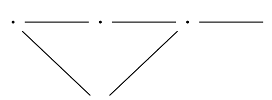
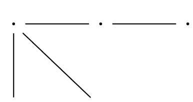


Another is



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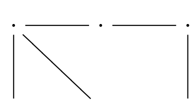
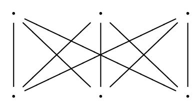


Another is



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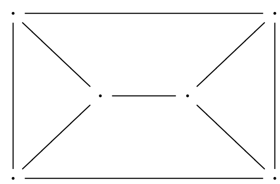


10 Let X be a random variable with values from $\{1, 2, \ldots, 100\}$ each occuring with probability $\mathbb{P}(X = n) = \frac{1}{100}$. Let Y be the total obtained by rolling 16 fair dice and taking the total. Y and X are independant. What is the probability that $Y \geqslant X$?

[Hint: how is this probability related to $\mathbb{E}(Y)$? Work out the probability conditional on a fixed value of Y first. You may assume that the expectation of a single roll of a fair die is $\frac{7}{2}$.]

The probability that $Y \geqslant X$ given that $Y = y$ where y is an integer between 1 and 100 inclusive (the values that Y can take are integers between 16 and 96, so they are of this form) is $\frac{y}{100}$, since there are 100 possible (equally likely) values for X, and y of them are less than or equal to y. Therefore, the probability that $Y \geqslant X$ is

$$\sum_{y=1}^{100} \mathbb{P}(Y = y)\frac{y}{100}$$

This is

$$\frac{1}{100}\sum_{y=1}^{100} y\mathbb{P}(Y = y) = \frac{1}{100}\mathbb{E}(Y)$$

We can calculate the expectation of Y because it is a sum of random variables whose expectations we can calculate – Let Y_i be the result of the ith roll. Then $Y = \sum_{i=1}^{16} Y_i$, so

$$\mathbb{E}(Y) = \sum_{i=1}^{16} \mathbb{E}(Y_i) = \sum_{i=1}^{16} \frac{7}{2} = 56$$

Therefore, $\mathbb{P}(Y \geqslant X) = \frac{56}{100} = \frac{14}{25}$.

[Note that it does not matter whether the dice are independant or not. The same argument shows that the probability is $\frac{14}{25}$ anyway. It is however necessary for Y and X to be independant.]

11 *One person in 100 has a certain disease. There is a test, which when given to a person with the disease has a 90% (0.9) probability of giving a positive result. When given to a person without the disease, it has a 2% (0.02) probability of giving a positive result. A person is chosen at random and tested.*

(a) What is the probability that the test gives a positive result?

The probability that the person has the disease, and the test gives a positive result is $0.01 \times 0.9 = 0.009$. The probability that the person does not have the disease, but the test still gives a positive result is $0.99 \times 0.02 = 0.0198$. Therefore, the overall probability that the test gives a positive result is 0.0288.

(b) Given that the test gives a positive result, what is the probability that the person actually has the disease?

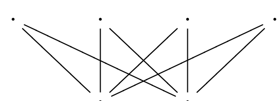
The probability that the person has the disease given that the test was positive is

$$\frac{\mathbb{P}(\text{Person has disease and test was positive})}{\mathbb{P}(\text{test was positive})}$$

which is $\frac{0.009}{0.0288} = \frac{90}{288} = \frac{5}{16}$

12 *2-colour the edges of a K_7 independantly red with probability $\frac{1}{2}$ and blue with probability $\frac{1}{2}$.*

(a) What is the expected number of monochromatic $K_{4,2}$ (complete bipartite graphs on sets of 4 and 2 vertices) – e.g.



A $K_{4,2}$ has 8 edges, so the probability that it is monochromatic is $\frac{1}{2^7}$. (There are 2^8 equally likely colourings of the $K_{4,2}$; it is only monochromatic in two of them.) There are $\binom{7}{4,2} = \binom{7}{2}\binom{5}{4} = 105$ $K_{4,2}$s (choose the set of two vertices and the set of 4 vertices) so the expected number of monochromatic $K_{4,2}$s is $\frac{105}{128}$.

(b) Deduce that it is possible to 2-colour a K_7 without a monochromatic $K_{4,2}$.

Since the number of monochromatic $K_{4,2}$ is either 0 or at least 1, the probability that it is at least 1 is at most the expected number of $K_{4,2}$, so this probability is less than one. However, if every colouring had a $K_{4,2}$, the probability would be 1. Therefore, there must be some 2-colouring of a K_7 without a monochromatic $K_{4,2}$.