MATH 2113/CSCI 2113, Discrete Structures II
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Homework Sheet 2
Hints & Model Solutions

Sheet 1

6 *Suppose we have a cube made of $n \times n \times n$ smaller cubes ($n \geqslant 2$). We call a line through the cube a set of n of the smaller cubes such that the centres of all the smaller cubes are in a straight line. How many lines are there through the cube?*

Solution 1 Imagine that our cube is the middle cube of an $(n+2) \times (n+2) \times (n+2)$ cube. We extend our line to a line of the outer cube. There is exactly one way to do this, since our line already goes through at least two points, so its direction is fixed.

The line goes through two of the new cubes. On the other hand, given one of the new cubes, there is exactly one line through the inner cube that goes through this cube. Therefore, the number of lines is half the number of new cubes – i.e. $\frac{(n+2)^3-n^3}{2}$ line ($= \frac{6n^2+12n+8}{2} = 3n^2+6n+4$).

Solution 2 There are 13 possible directions for our line – suppose that the edges of our cube are along the x, y and z axis. First we consider directed lines. Then each of the x, y and z coordinates will be increasing, decreasing or constant. This gives $3 \times 3 \times 3 = 27$ possibilities. However, x, y and z cannot all be constant, since that would not give a line. Therefore, there are 26 possible directions for lines, but since our lines are not directed, we consider one direction the same as the opposite direction, so there are only 13 possible directions for lines.

3 of these directions have 2 coordinates constant. Lines in these directions can start at any of the n^2 cubes with the appropriate coordinate equal to 0. 6 of the directions have 1 coordinate constant, and they can start at any of the n cubes on an edge – e.g. the line with increasing x coordinate, decreasing y coordinate and constant z coordinate can start at any of the points with $x = 0, y = n$. The final 4 directions are the diagonals of the cube, and there is only one line in each such direction. Therefore, the total number of lines is $3n^2 + 6n + 4$ ($= \frac{(n+2)^3-n^3}{2}$).

Compulsory questions

1 *Show that* $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

First we note that $\binom{n}{k} = \binom{n}{n-k}$, since choosing a set is equivalent to choosing its complement. Now by the multiplication rule, $\binom{n}{k}\binom{n}{n-k}$ is the number of ways of choosing k elements from the first n elements of a $2n$-element set, then choosing $n-k$ elements from the last n elements of this $2n$-element set.

When we choose n elements from a $2n$-element set X, the number of these elements that lie among the first n elements of X is some k with $0 \leqslant k \leqslant n$, and the number of elements that lie among the last n elements is $n-k$. Therefore, by the addition rule, the number of ways of choosing n elements from a set of $2n$ elements is the sum over k from 0 to n of the number of ways of choosing these n elements so that exactly k are in the first n elements of X. This sum is exactly the RHS of the equation we want to prove.

2 *(a) Show that* $\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}$.

Suppose we have a set X of n elements. The LHS is the number of ways of choosing disjoint subsets A and B of X such that A has a elements and B has b elements. However, a different way to choose these sets is to first choose the union $A \cup B$, then to choose A. B will then be chosen automatically as $A \cup B \setminus A$. We can choose the union in $\binom{n}{a+b}$ different ways, and once we have chosen it, we can choose A in $\binom{a+b}{a}$ different ways, so by this method, there are $\binom{n}{a+b}\binom{a+b}{a}$ ways to choose A and B. This is the RHS. But the number of ways to choose A and B does not depend on the way in which we choose them, so the LHS and RHS must be equal.

(b) What is $\sum_{k=1}^{n}\binom{n}{k}k^2$*? [Hint:* $k^2 = 2\binom{k}{2} + \binom{k}{1}$*.]*

$\sum_{k=1}^{n}\binom{n}{k}k^2 = \sum_{k=0}^{n} 2\binom{n}{k}\binom{k}{2} + \binom{n}{k}\binom{k}{1}$. By part (a), this is $\left(\sum_{k=2}^{n} 2\binom{n}{2}\binom{n-2}{k-2}\right) + \left(\sum_{k=1}^{n}\binom{n}{1}\binom{n-1}{k-1}\right) = 2\binom{n}{2}2^{n-2} + \binom{n}{1}2^{n-1} = 2^{n-1}\left(\binom{n}{2} + \binom{n}{1}\right) = 2^{n-1}\binom{n+1}{2}$.

3 *How many subsets of* $\{1, 2, \ldots, 17\}$ *contain at most two multiples of 3?*

The set $\{1, 2, \ldots, 17\}$ contains 5 multiples of 3 (3, 6, 9, 12, and 15), and $17 - 5 = 12$ non-multiples of 3. We can express a subset of $\{1, 2, \ldots, 17\}$ as a pair consisting of a subset of $\{3, 6, 9, 12, 15\}$ and a subset of the complement: $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}$. The condition that it should contain at most 2 multiples of 3 means that the first subset should have at most 2 elements. There are 16 subsets of a set of size 5 that have at most 2 elements – We can either work this out as $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} = 16$, or we can notice that exactly one of the subset and its complement has at most two elements, so there are twice as many subsets of a 5-element set as there are subsets with at most two elements: divide the collection of all subsets of a 5-element set into complementary pairs, one of each such pair will have size at most two. There are 32 subsets of a 5-element set, so there are 16 subsets with size at most two.

For each of these 16 subsets of $\{3, 6, 9, 12, 15\}$, we can choose any of the $2^{12} = 4,096$ subsets of the 12 non-multiples of 3. This gives a total of

$16 \times 2^{12} = 2^{16} = 65,536$ subsets of $\{1, 2, \ldots, 17\}$ that contain at most 2 multiples of 3.

4 *(a) How many solutions are there to $x_1 + x_2 + x_3 + x_4 = 18$ where x_1, x_2, x_3 and x_4 are natural numbers ($\{0, 1, 2, 3, \ldots\}$)?*

We can represent a solution to this equation by drawing x_1 dots, then drawing a vertical line, then drawing x_2 dots, then another vertical line, then x_3 dots, then another vertical line, then finally x_4 dots. In total, there will be 18 dots and 3 vertical lines. The vertical lines can be in any 3 positions – given 3 positions for the vertical lines, x_1 is the number of dots before the first, x_2 is the number of dots between the first and second vertical lines, x_3 is the number of dots between the second and third vertical lines, and x_4 is the number of dots to the right of the third vertical line.

There are 21 symbols in total, and 3 of them are vertical lines, so there are $\binom{21}{3}$ possible ways of placing the vertical lines. Therefore there are $\binom{21}{3}$ (=1330) solutions.

(b) How many solutions are there to $x_1 + 2x_2 + 3x_3 = 10$ for x_1, x_2 and x_3 natural numbers?

For this question, an approach like the one in part (a) cannot be applied, so we have to solve it directly by looking at the possibilities:

First we observe that $3x_3 \leqslant 10$, so x_3 is at most 3. If $x_3 = 0$, then x_2 can be anything from 0 to 5, and then x_1 is fixed, so there are 6 solutions with $x_3 = 0$. If $x_3 = 1$, we need $x_1 + 2x_2 = 7$, so x_2 can be anything from 0 to 3, and then x_1 is fixed, so there are 4 solutions with $x_3 = 1$. If $x_3 = 2$, x_2 can be anything from 0 to 2, so there are 3 solutions. Finally if $x_3 = 3$, then x_2 must be 0, so there is only 1 solution with $x_3 = 3$.

Therefore, by the addition rule, there are $6 + 4 + 3 + 1 = 14$ solutions.

5 *(a) In a class with 13 students, there are 5 mathematicians and 8 computer scientists. How many subsets of the students in the class contain the same number of mathematicians and computer scientists? [Hint: The easy way to answer this question involves considering a different set from the set to be chosen (but related to it).]*

Solution 1: The number of mathematicians can be anything from 0 to 5. The number of subsets with i mathematicians and i computer scientists is $\binom{5}{i}\binom{8}{i}$. Therefore, the number of subsets with the same number of mathematicians and computer scientists is $\binom{5}{0}\binom{8}{0}+\binom{5}{1}\binom{8}{1}+\binom{5}{2}\binom{8}{2}+\binom{5}{3}\binom{8}{3}+\binom{5}{4}\binom{8}{4}+\binom{5}{5}\binom{8}{5}$. (This is $1\times1+5\times8+10\times28+10\times56+5\times70+1\times56 = 1287$. In fact, we can factor, $1287 = 9 \times 143 = 9 \times 11 \times 13 = \binom{13}{5}$.)

Solution 2: Instead of the subset A containing the same number of mathematicians and computer scientists, consider the set A' containing all the computer scientists in A, and all the mathematicians **not** in A. The number of mathematicians in A' is 5 minus the number of mathematicians in

A. The number of mathematicians in A is the same as the number of computer scientists in A, which is also the number of computer scientists in A'. Therefore, the total number of students in A' is 5. On the other hand, given a subset B containing 5 students, there are as many computer scientists in B as there are mathematicians not in B, so any subset B of 5 students occurs as A' for some subset A with the same number of mathematicans and computer scientists.

Therefore, the number of subsets with the same number of mathematicians and computer scientists is the same as the number of subsets with 5 members, so there are $\binom{13}{5}$ such subsets (or 1287).

Bonus question

(b) How many contain at least as many mathematicians as computer scientists? [Hint: This question will be a lot easier if you can find a simple explanation for why the solution to part (a) is what it is.]

Following Solution 2 to part (a), we let A be the subset chosen, and consider the set A' of computer scientists in A and mathematicians not in A. A' has at most 5 members, and given a subset with at most 5 members, it occurs as such an A', so the total number of such sets is $\binom{13}{0}+\binom{13}{1}+\binom{13}{2}+\binom{13}{3}+\binom{13}{4}+\binom{13}{5}$. (This is $1+13+78+286+715+1287 = 2380$.)