MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Homework Sheet 5
Hints & Model Solutions

Compulsory questions

1 *Let a_n be defined recursively by $a_n = \sum_{i=1}^{n} \frac{a_{n-i}(-1)^{i-1}}{i!}$ for $n \geqslant 1$, and $a_0 = 1$.*

(a) Find the generating function for a_n.

The generating function is given by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We multiply the recurrence by x^n and sum over n to get:

$$f(x) - 1 = \sum_{1 \leqslant i \leqslant n} \frac{a_{n-i}(-1)^{i-1}x^n}{i!} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} \frac{a_m(-1)^{i-1}x^{m+i}}{i!} =$$
$$-\sum_{m=0}^{\infty} a_m x^m \sum i = 1^{\infty} \frac{(-x)^i}{i!} = (1 - e^{-x}) \sum_{m=0}^{\infty} a_m x^m =$$
$$(1 - e^{-x})f(x)$$

This gives $f(x)e^{-x} = 1$, or $f(x) = e^x$.

(b) Find a general formula for a_n.

From this it follows immediately that $a_n = \frac{1}{n!}$.

2 *(a) Find a recurrence relation for the number of ways to tile a $3 \times n$ chessboard with 3×1 blocks.*

Let T_n be the number of tilings. We can either start with a vertical block and a tiling of the remaining $3 \times (n-1)$ chessboard, or a column of 3 horizontal blocks and a tiling of the remaining $3 \times (n-3)$ chessboard. Therefore, T_n satisfies the recurrence $T_n = T_{n-1} + T_{n-3}$, where we take $T_0 = 1$ and $T_n = 0$ for all $n < 0$.

(b) Find a recurrence relation for the number of ways to tile a $3 \times n$ chessboard with 3×1 and 3×2 blocks.

Let T_n be the number of tilings. We can start with a vertical 3×1 block and a tiling of the remaining $3 \times (n-1)$ chessboard, or a vertical 3×2 block and a tiling of the remaining $3 \times (n-2)$ chessboard, or a column of horizontal blocks, for which we have 3 choices - $(1,1,1)$, $(2,1)$, and $(1,2)$, and a tiling of the remaining $3 \times (n-3)$ chessboard. Therefore, T_n satisfies the recurrence $T_n = T_{n-1} + T_{n-2} + 3T_{n-3}$, where we take $T_0 = 1$ and $T_n = 0$ for all $n < 0$.

3 We have a sequence of n tiles that we want to paint with 4 colours: red, green, blue and yellow, in such a way that no two adjacent tiles are the same colour, and no two adjacent pairs of tile have the same colours - so for example RGBGB would not be a valid colouring, since the sequence GB is repeated.

(a) Find a recurrence relation for the number of valid colourings.

Hint: Divide the collection of valid colourings into two types – those in which the nth and $n-2$th tiles are different colours, and those in which they are the same colour. The number of colourings of each type can be calculated in terms of the total number of colourings with fewer tiles. Your recurrence might not work for small values of n.

(b) Solve the recurrence relation.

Hint: It is possible to get a second order, constant coefficient, homogeneous linear recurrence. The solution to these was covered in course 2112, and is in Chapter 8.3 of the textbook. Alternatively, you can use generating functions.

4 Define the sequence a_n recursively by $a_0 = 0$ and $a_{n+1} = 2a_n + (n+1)^2$ for $n \geqslant 0$.

(a) Find the generating function for a_n.

Multiply both sides of the recurrence by x^n to get $a_{n+1}x^n = 2a_n x^n + (n+1)^2 x^n$, then sum over n to get

$$\sum_{m=1}^{\infty} a_m x^{m-1} = \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} (n+1)^2 x^n$$

If we let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^n$ (the last equality is because $a_0 = 0$) then we have

$$\frac{f(x)}{x} = 2f(x) + \sum_{n=0}^{\infty} (n+1)^2 x^n$$

We evaluate the last sum by differentiating the taylor series for $\frac{1}{1-x}$ to get $\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$, so $\sum_{n=0}^{\infty}(n+1)x^{n+1} = \frac{x}{(1-x)^2}$ and $sum_{n=0}^{\infty}(n+1)^2 x^n = \frac{(1-x)+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$. We therefore get

$$f(x)\left(\frac{1}{x} - 2\right) = \frac{1+x}{(1-x)^3}$$

or $f(x) = \frac{x(1+x)}{(1-2x)(1-x)^3}$.

(b) Expand the generating function as a partial fraction.

The general form will be

$$\frac{x(1+x)}{(1-2x)(1-x)^3} = \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{1-2x}$$

We multiply through by $(1-x)^2(1-2x)$, and substitute $x=1$ to get $A=-2$ and $x=\frac{1}{2}$ to get $D=6$. We substitute these values to get

$$\begin{aligned}
x(1+x) &= -2(1-2x)+6(1-x)^3+B(1-x)(1-2x)+C(1-x)^2(1-2x) \\
6x^3-17x^2+15x-4 &= B(1-x)(1-2x)+C(1-x)^2(1-2x) \\
(1-x)(-4+11x-6x^2) &= B(1-x)(1-2x)+C(1-x)^2(1-2x) \\
(1-x)(1-2x)(3x-4) &= (B+C(1-x))(1-x)(1-2x)
\end{aligned}$$

Thus we get $C=-3$, $B=-1$, so

$$f(x)=\frac{6}{1-2x}-\frac{3}{1-x}-\frac{1}{(1-x)^2}-\frac{2}{(1-x)^3}$$

(c) Use this to find the a_n.

We know that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$, $\frac{1}{(1-x)^3}=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$, and $\frac{1}{1-2x}=\sum_{n=0}^{\infty}2^nx^n$. This gives

$$a_n=6\times 2^n-3-(n+1)-2\frac{(n+1)(n+2)}{2}=6\times 2^n-3-(n+1)(n+3)$$