MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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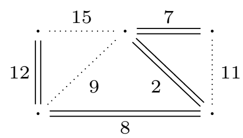
Homework Sheet 8
Due: Wednesday 26th March: 1:30 PM

Compulsory questions

1 *Find minimal spanning trees for the following graphs:*

(a)



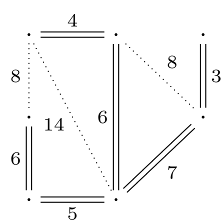
Using Kruskal's algorithm:

We first include the edges of lengths 2, 7 and 8. The next edge we try is the edge of length 9. This forms a cycle, so we don't include it. Next we try the edge of length 11. This forms another cycle, so we don't include it. The edge of length 12 does not form a cycle, so we include it. Finally, the edge of length 15 forms a cycle, so we don't include it.

Using Prim's algorithm:

We start with the top-left vertex. The shortest edge from this vertex is the one of length 12, so we add that one. The shortest edge we can choose now is the one of length 8, so we choose that. The shortest edge we can now select is the one of length 2. Finally, the shortest edge we can select is the one of length 7.

(b)



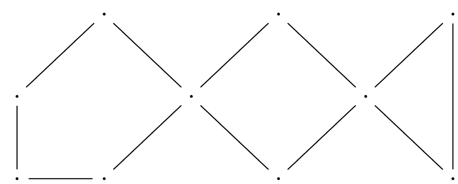
Using Kruskal's algorithm:

In this case, the shortest edges are the ones of lengths 3, 4, 5, 6, 6, and 7. These do not form any cycles, so we add them all. Now all the remaining edges do not form cycles, so we do not add any of them.

Using Prim's algorithm:

We start with the top-left vertex. The shortest edge from this vertex is the one of length 4, so we add that one. The shortest edge we can choose now is the one of length 6, so we choose that. The shortest edge we can now select is the one of length 5. Now the shortest edge we can select is the one of length 6. Next we select the edge of length 7, and finally, we select the edge of length 3.

2 *How many spanning trees are there for the following graph?*

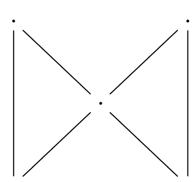


A spanning tree for this graph has no cycles, so we need to remove one edge from each of the cycles. There are 5 choices for the edge to remove from the first cycle, 4 from the second and 3 from the third. This gives $5 \times 4 \times 3 = 60$ spanning trees.

3 *(a) What is the minimum number of cycles that a graph with 5 vertices and 6 edges can have? Justify your answer.*

An acyclic graph with 5 vertices can have at most 4 edges. Therefore, if G is a graph with 5 vertices and 6 edges, let G' be a maximal acyclic subgraph (i.e. a spanning tree if G is connected). G has at least 2 edges that are not in G'. Let e and f be two edges in G and not in G'. Since G' is maximal acyclic, there are cycles in $G' \cup e$ and $G' \cup f$. These must be different cycles since the first must contain e and the second must contain f. Therefore, G must contain at least two cycles.

(b) Draw a graph with 5 vertices, 6 edges and the number of cycles in your answer to (a). [Careful – it's easy to draw a graph with too many cycles.]

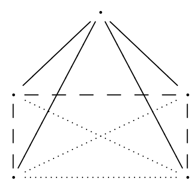


is one possibility.

4 *(a) How many trees are needed to cover all the edges of K_5 (the complete graph on 5 vertices)? i.e. we want a collection of trees with vertices chosen from the vertices of our K_5, such that the union of their edges is the collection of all edges of the K_5. Justify your answer.*

A tree with at most 5 vertices has at most 4 edges, while K_5 has $\binom{5}{2} = 10$ edges, so we need at least 3 trees to cover all the edges.

(b) Draw a collection of this many trees that cover the edges of the K_5.



5 *Let T be a tree with $n \geqslant 3$ vertices. Let G be a graph obtained by adding one new edge to T. Show that G contains exactly one cycle.*

We showed that a tree is maximal acyclic, so G must contain a cycle. Suppose that G contains 2 cycles. They must both contain the new edge, since otherwise, they would be cycles in T. Let these new cycles be $v_0xv_1e_1v_2e_2\dots v_me_mv_0$ and $v_0xv_1f_1w_2f_2\dots w_{n-1}f_nv_0$, where x is the new edge and $e_1,\dots,e_m,f_1,\dots,f_n$ are edges of T. Now if none of the v_i and w_j are equal, then $v_1e_1v_2\dots v_me_mv_0f_nw_{n-1}\dots w_1f_1v_1$ is a cycle in T, which is impossible. Similarly, if v_i is the first vertex in the first cycle (except v_1 and v_2) to also be in the second cycle, and $v_i = w_j$, then $v_1e_1v_2\dots e_{i-1}v_if_{j-1}w_{j-1}\dots w_2f_1v_1$ is a cycle in T unless $i = j = 1$. If $i = j = 1$, we can look at the next $v_i = w_j$, and a cycle based on that.