MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Homework Sheet 9
Hints & Model Solutions

Compulsory questions

1 *(a) Show that if we 3-colour the complete graph on 17 vertices, we get a monochromatic triangle.*

Pick a vertex. It has 16 neighbours, so by the pigeon-hole principle, some 6 must be joined to it by edges of the same colour, WLOG red. If any edge between two of these vertices is red, it completes a red triangle. However, if all edges between these 6 are either blue or green, then there is a monochromatic triangle among these 6 vertices, since $R(3,3) = 6$.

(b) Is it possible to partition the set $\{1, 2, \ldots, 16\}$ into 3 sets such that no set contains any number which is the sum of two numbers in the same set (possibly the same number twice)? [Hint: Given such a partition into the sets A, B, and C, take a complete graph on 17 vertices labelled $v_1, v_2, \ldots, v_{17}$, and colour the edge v_iv_j red if the difference $|i-j|$ is in A, blue if $|i-j|$ is in B, and green if $|i-j|$ is in C. What does a monochromatic triangle mean for this colouring?]

It is not possible – Colour the graph as suggested. By Part (a), there is a monochromatic triangle. This means there are 3 numbers $x \leqslant y \leqslant z$ in the set $\{1, \ldots, 17\}$ such that $y-x$, $z-y$ and $z-x$ are all in the same set. However, $z-x = z-y+y-x$, so we have a number expressed as a sum of two numbers in the same set.

2 *Suppose we colour each edge of the complete graph on 11 vertices red with probability $\frac{1}{3}$ and blue with probability $\frac{2}{3}$ (so it is always coloured either red or blue).*

(a) What is the expected number of red K_4s?

The probability that a given K_4 is red is $\left(\frac{1}{3}\right)^6$. There are a total of $\binom{11}{4}$ K_4s in the graph, so the expected number of red K_4s is $\binom{11}{4}\left(\frac{1}{3}\right)^6$.

(b) What is the expected number of blue K_6s?

The probability that a given K_6 is blue is $\left(\frac{2}{3}\right)^{15}$. There are a total of $\binom{11}{6}$ K_6s in the graph, so the expected number of blue K_6s is $\binom{11}{6}\left(\frac{2}{3}\right)^{15}$.

(c) Deduce that there is a 2-colouring of the complete graph on 11 vertices without a red K_4 or a blue K_6. [Hint: $2^{14} < 3^9$, $\binom{11}{4} = 330$, $\binom{11}{6} = 396$, $3^6 = 729$.]

Due to an error in the calculation, this question was removed from the homework. It is possible to construct examples by hand. For example, we can form a 2-colouring of a graph on 15 vertices with no blue K_6 or red K_4 – simply divide the vertices into 3 sets A_1, A_2 and A_3 of 5 vertices each, and colour an edge blue if it goes between two vertices in the same A_i, red if it goes between vertices in different A_i.

[This is far from best – there are in fact colourings of the edges of the complete graph on 17 vertices with no red or blue K_4.]

3 *(a) Show that if we 2-colour (red and blue) the edges of the complete graph on 10 vertices, we get either a red triangle or a blue complete graph on 4 vertices.*

Consider a vertex. Since it has 9 neighbours, it must have either 4 red neighbours or 6 blue neighbours. If it has 4 red neighbours, any red edge between two of them would complete a red triangle. On the other hand, if there are no red edges between its 4 red neighbours, they form a blue K_4.

If it has 6 blue neighbours, then the subgraph on these neighbours contains either a red triangle or a blue triangle. If it contains a red triangle, then the larger graph contains a red triangle. If it contains a blue triangle, then with the original vertex, this forms a blue K_4 in the larger graph.

Bonus Question

(b) Show that if we 2-colour (red and blue) the edges of the complete graph on 9 vertices, we get either a red triangle or a blue complete graph on 4 vertices. [Hint: In part (a), you probably found a condition on a vertex that would force the existence of either a red triangle or a blue K_4. With 10 vertices, this condition must hold for every vertex. With 9 vertices, it doesn't need to hold for every vertex, but suppose it doesn't hold for any vertex, and consider the subgraph consisting of just the blue edges.]

As in part (a), if there is a vertex with either 6 blue neighbours or 4 red neighbours, then we get either a blue K_4 or a red triangle. Therefore, in order not to have either of these, we must have that each vertex has 5 blue neighbours and 3 red neighbours. Now consider the subgraph formed by the blue edges. It has 9 vertices, all of degree 5. This is impossible as $9 \times 5 = 45$ is odd, so cannot be twice the number of edges in a graph. Therefore, by contradiction, the graph has some vertex with 6 blue neighbours or 5 red neighbours. This leads to it having a red triangle or a blue K_4.