MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
Toby Kenney
Midterm Examination

Calculators not permitted. Answers may be left in reasonably simplified forms – e.g. binomial coefficients, factorials, etc. Justify all your answers.

Section A – Wednesday 5th March 12:35-1:20PM

1 *How many subsets of* $\{1, 2, \ldots, 19\}$ *contain twice as many odd numbers as even numbers?*

$\{1, 2, \ldots, 19\}$ has 10 odd numbers and 9 even numbers, so if a subset has twice as many odd numbers as even numbers, there are the following possibilities:

Odd numbers	Even numbers	Sets	[=]
0	0	$\binom{10}{0}\binom{9}{0}$	1
2	1	$\binom{10}{2}\binom{9}{1}$	405
4	2	$\binom{10}{4}\binom{9}{2}$	7560
6	3	$\binom{10}{6}\binom{9}{3}$	17640
8	4	$\binom{10}{8}\binom{9}{4}$	567
10	5	$\binom{10}{10}\binom{9}{5}$	126

Therefore the total number of such sets is $\binom{10}{0}\binom{9}{0} + \binom{10}{2}\binom{9}{1} + \binom{10}{4}\binom{9}{2} + \binom{10}{6}\binom{9}{3} + \binom{10}{8}\binom{9}{4} + \binom{10}{10}\binom{9}{5}$. [Which is 26,299.]

2 *250 students sit a multiple choice exam with n questions, each of which has 4 choices. The students know that no three consecutive questions all have the same answer, so no student submits a set of solutions with 3 consecutive answers the same.*

(a) Let a_n be the number of possible solutions to the exam. Find a recurrence relation satisfied by the a_n.

The number of solutions in which the last two answers are different is $3a_{n-1}$. (There are 3 choices for the last answer – the 3 not used for the second last answer.) The number of solutions in which the last two answers are the same is $3a_{n-2}$ (There are 3 choices for the last two answers – the three not used for the third last answer.) Therefore, $a_n = 3a_{n-1} + 3a_{n-2}$, for $n \geqslant 3$, $a_1 = 4$, $a_2 = 16$.

(b) If no two students hand in identical sets of solutions, what is the smallest possible value of n? [The answer is small enough that you do not have to find a general solution to the recurrence from (a).]

We use the recurrence in (a) to find the first few a_n: the sequence begins $4, 16, 60, 228, 864, \ldots$. The first term with $a_n > 250$ is $n = 5$. Therefore, if $n < 5$, the number of possible solutions is at most 228, so by the pigeon-hole principle, some two students must submit identical solutions. Therefore, if all students submit different solutions, then the smallest possible value of n is 5.

3 How many numbers between 1008 and 2008 inclusive are multiples of 2, 3, or 7?

We use the inclusion-exclusion principle. Let A_2 be the set of multiples of 2 between 1008 and 2008 inclusive. Let A_3 be the set of multiples of 3 between 1008 and 2008 inclusive. Let A_7 be the set of multiples of 7 between 1008 and 2008 inclusive. We want to find $|A_2 \cup A_3 \cup A_7|$. By the inclusion-exclusion principle, this is equal to $|A_2|+|A_3|+|A_7|-|A_2 \cap A_3|-|A_2 \cap A_7|-|A_3 \cap A_7|+|A_2 \cap A_3 \cap A_7|$. $|A_2| = 501$ $(1004-503)$ $|A_3| = 334$, $(669-335)$ $|A_7| = 142$, $(286-143)$ $|A_2 \cap A_3| = 167$, $(334-167)$ $|A_2 \cap A_7| = 72$, $(143-71)$ $|A_3 \cap A_7| = 48$, $(95-47)$ $|A_2 \cap A_3 \cap A_7| = 24$, $(47-23)$. Therefore, $|A_2 \cup A_3 \cup A_7| = 501 + 334 + 142 - 167 - 72 - 48 + 24 = 714$. There are 714 multiples of 2, 3, or 7 between 1008 and 2008 inclusive.

4 I have 3 urns. The first contains 3 blue balls, 2 red balls and 1 yellow ball. The second contains 7 blue balls and 1 red ball. The third contains 1 blue ball and 4 red balls.

(a) I pick an urn uniformly at random, and pick a ball from it uniformly at random. It is blue. What is the probability that it came from the first urn?

The probability of picking a blue ball from the first urn is $\frac{1}{3} \times \frac{3}{6} = \frac{1}{6}$ (probability of picking the first urn times probability of picking a blue ball from it). The probability of picking a blue ball from any urn is $\frac{1}{3} \times \frac{3}{6} + \frac{1}{3} \times \frac{7}{8} + \frac{1}{3} \times \frac{1}{5} = \frac{63}{120} = \frac{21}{40}$. Therefore, the probability that a blue ball was chosen from the first urn is $\frac{\frac{1}{6}}{\frac{21}{40}} = \frac{40}{6 \times 21} = \frac{20}{63}$.

(b) I pick another ball from the same urn without replacement. What is the probability that it is yellow?

From the first part, the probability that the first ball came from the first urn is $\frac{20}{63}$. The probability that the second ball is yellow is therefore $\frac{20}{63} \times \frac{1}{5} = \frac{4}{63}$.

Section B – Friday 7th March 12:35-1:20PM

1 *A gambler starts with \$12. He makes a series of bets of \$1 on the roll of a fair die. If the roll is 6, he gets his original dollar back, and a further \$4. If it is anything else, he loses his \$1.*

(a) Let X_n be the amount the gambler gains on the nth bet that he makes. What is $\mathbb{E}(X_n)$?

$X_n = 4$ with probability $\frac{1}{6}$, and $X_n = -1$ with probability $\frac{5}{6}$, so $\mathbb{E}(X_n) = \frac{1}{6} \times 4 - \frac{5}{6} \times 1 = -\frac{1}{6}$.

(b) What is the expected amount of money that the gambler has after n rolls?

After n rolls, the amount of money the gambler has is $12 + X_1 + X_2 + \cdots + X_n$, so the expected amount of money the gambler has is

$$\mathbb{E}(12+X_1+\cdots+X_n) = 12+\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_n) = 12-\frac{1}{6}-\cdots-\frac{1}{6} = 12-\frac{n}{6}$$

(c) The gambler continues playing for 12 rolls. Let p be the probability that he ends up with at least \$20. Show that $p \leqslant \frac{1}{2}$. Explain your reasoning carefully.

After 12 rolls, the expected amount of money the gambler has is \$10. The least money the gambler can have is \$0. The expected amount of money the gambler has is at least $20p + 0(1-p) = 20p$, so we get that $10 \geqslant 20p$, or $p \leqslant \frac{1}{2}$.

[Actually, in order to have at least \$20, he must win at least 4 times, so $p = \sum_{n=4}^{12} \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{12-n} \binom{12}{n}$ or approximately $\frac{1}{8}$.]

2 *Define a recurrence by $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$.*

(a) Show that the generating function for the sequence a_n is

$$A(x) = \frac{1}{(1-3x)(1-2x)}$$

Let the generating function be $A(x) = \sum_{n=0}^{\infty} a_n x^n$. We multiply the recurrence by x^n and sum from $n = 1$ to ∞ to get

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 2^n x^n$$

The first sum is $A(x) - a_0 = A(x) - 1$. The second sum is $xA(x)$, and the third sum is $\frac{1}{1-2x} - 1$, so the equation becomes $A(x)(1-3x) = \frac{1}{1-2x}$ or

$$A(x) = \frac{1}{(1-3x)(1-2x)}$$

(b) Find a general formula for a_n.

We can express $A(x)$ as a sum of fractions $A(x) = \frac{B}{1-3x} + \frac{C}{1-2x}$. Multiplying through by $(1-2x)(1-3x)$, we get $B(1-2x) + C(1-3x) = 1$ Substituting $x = \frac{1}{2}$ gives $-\frac{1}{2}C = 1$ or $C = -2$. Substituting $x = \frac{1}{3}$ gives $\frac{1}{3}B = 1$ or $B = 3$. We therefore get $A(x) = \frac{3}{1-3x} - \frac{2}{1-2x} = 3\sum_{n=0}^{\infty} 3^n x^n - 2\sum_{n=0}^{\infty} 2^n x^n$, or

$$a_n = 3^{n+1} - 2^{n+1}$$

3 *How many 4-digit numbers are there with the digits in increasing order? [Not necessarily strictly increasing order, so 1147 and 3556 are OK. Leading zeros are not permitted.]*

solution 1: Add 1 to the second digit, 2 to the third digit and 3 to the fourth digit. Now the digits range from 1 to 12, and all 4 digits are different, so we are just considering subsets of $\{1, \ldots, 12\}$ of size 4, and there are $\binom{12}{4}$ of these $(= 495)$.

solution 2: Use an 'x' to denote each of the digits $\{2, 3, \ldots, 9\}$, and use vertical lines to represent the digits of the number – for each 1 in the number, put a vertical line to the left of all the 'x's, and for each other digit, put the vertical line to the right of the 'x' corresponding to that digit. For example, the number 1377 would be represented by the sequence of symbols "|xx|xxxx||xx". There are a total of 12 symbols in each sequence, and 4 of them are vertical lines. Therefore, the number of such sequences is $\binom{12}{4}$.

4 *n fair dice are rolled. What is the probability that the highest number rolled is a 4?*

There are 4^n rolls with no number higher than a 4, and 3^n rolls with no number higher than a 3. If 4 is the highest number rolled, then there is no number higher than a 4, but there is a number higher than a 3, so the number of rolls where 4 is the highest number is $4^n - 3^n$, and the probability of this is $\frac{4^n - 3^n}{6^n}$.

Section C – Thursday 4th March. Makeup exam.

1 *I have 3 fair dice: a red die whose faces are numbered 3,3,3,3,3,6; a green die whose faces are numbered 2,2,2,5,5,5; and a blue die whose faces are numbered 1,4,4,4,4,4.*

Two players, A and B each pick a die and roll it. Whoever gets the higher number wins.

(a) What is the probability that Player A wins if:

(i) Player A chooses the red die and player B chooses the green die?

If Player A rolls a 6, he wins. If he rolls a 3, he has a $\frac{1}{2}$ chance of winning. Therefore his chance of winning is $\frac{1}{6} \times 1 + \frac{5}{6} \times \frac{1}{2} = \frac{7}{12}$.

(ii) Player A chooses the green die and player B chooses the blue die?

If Player A rolls a 5, he wins. If he rolls a 2, then he wins if player B rolls a 1. Therefore, his total chance of winning is $\frac{1}{2} + \frac{1}{2} \times \frac{1}{6} = \frac{7}{12}$.

(iii) Player A chooses the blue die and player B chooses the red die?

Player A wins if and only if he rolls a 4 and player B rolls a 3. The probability of this is $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$.

(b) Is it better to choose the die first or second?

It is better to choose second, because whichever die is chosen first, there is a die that has a better than 0.5 chance of winning against it.

2 *I toss 5 (independant) fair coins. Which of the following sets of events are independant?*

(a) (i) The first two are both heads (ii) the third, fourth and fifth are all the same (iii) There are an even number of heads in the 5 tosses.

Call these events A, B, and C. $\mathbb{P}(A) = \frac{1}{4}$, $\mathbb{P}(B) = \frac{2}{8} = \frac{1}{4}$, $\mathbb{P}(C) = \frac{1}{2}$. $\mathbb{P}(A \cap B) = \frac{2}{32} = \frac{1}{16} = \frac{1}{4} \times \frac{1}{4}$, $\mathbb{P}(A \cap C) = \frac{4}{32} = \frac{1}{8} = \frac{1}{4} \times \frac{1}{2}$, $\mathbb{P}(B \cap C) = \frac{4}{32} = \frac{1}{8} = \frac{1}{4} \times \frac{1}{2}$, $\mathbb{P}(A \cap B \cap C) = \frac{1}{32} = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{2}$, so the events are independant.

(b) (i) There are at least 3 heads (ii) There are at least 3 tails.

Call these events A and B. $\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{2}$, but $\mathbb{P}(A \cap B) = 0$, so these events are not independant.

3 *Define a recurrence by $a_n = 2a_{n-1} + 7n + 3$, $a_0 = 0$.*

(a) Show that the generating function for the sequence a_n is

$$A(x) = \frac{10x - 3x^2}{(1-x)^2(1-2x)}$$

Let the generating function be $A(x) = \sum_{n=0}^{\infty} a_n x^n$. We multiply the recurrence by x^n and sum from $n = 1$ to ∞ to get

$$\sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + 7\sum_{n=1}^{\infty} nx^n + 3\sum_{n=1}^{\infty} x^n$$

The first sum is just $A(x) - a_0 = A(x)$, and the second sum is $xA(x)$. The third sum is $\frac{x}{(1-x)^2}$. (We can get this by differentiating the power series for $\frac{1}{1-x}$, for example.) Finally, the fourth sum is $\frac{1}{1-x} - 1 = \frac{x}{1-x}$. The equation is therefore $A(x)(1-2x) = \frac{7x}{(1-x)^2} + \frac{3x}{1-x} = \frac{10x-3x^2}{(1-x)^2}$, so we get

$$A(x) = \frac{10x - 3x^2}{(1-x)^2(1-2x)}$$

(b) Find a general formula for a_n.

We express the generating function $A(x) = \frac{10x-3x^2}{(1-x)^2(1-2x)}$ as a sum of fractions $A(x) = \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{1-2x}$. Multiplying through by $(1-x)^2(1-2x)$, we get $10x - 3x^2 = B(1-2x) + C(1-x)(1-2x) + D(1-x)^2$. [This holds for all x, even $x = 1$ and $x = \frac{1}{2}$, because when two polynomials agree almost everywhere, they must have the same coefficients, so they must actually agree everywhere.] Substituting $x = 1$, we get $7 = -B$, so $B = -7$. Substituting $x = \frac{1}{2}$, we get $\frac{17}{4} = \frac{D}{4}$, so $D = 17$. Substituting $x = 0$ gives $0 = B + C + D = 17 - 7 + C$, so $C = -10$. We therefore have

$$A(x) = \frac{17}{1-2x} - \frac{7}{(1-x)^2} - \frac{2}{1-x}$$

We know that the Taylor series for the first fraction is $17\sum_{n=0}^{\infty} 2^n x^n$, the second sum has Taylor series $-7\sum_{n=0}^{\infty} nx^{n-1} = -7\sum_{m=0}^{\infty}(m+1)x^m$, and the third has Taylor series $-10\sum_{n=0}^{\infty} x^n$. Therefore,

$$A(x) = \sum_{n=0}^{\infty}(17 \times 2^n - 7(n+1) - 10)x^n$$

Therefore, we have that

$$a_n = 17 \times 2^n - 7n - 17$$

4 *How many solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ with x_1, x_2, x_3, x_4, x_5 all natural numbers $\{0, 1, 2, \ldots\}$.*

Line up 16 'x's, and indicate a solution to the equation by putting 4 vertical bars between the 'x's, so that the line of 'x's is divided into 5 regions. We then get a solution by letting x_1 be the number of 'x's in the first region, x_2 be the number of 'x's in the second region, etc. For example, the solution $x_1 = 3, x_2 = 5, x_3 = 4, x_4 = 0, x_5 = 4$ would be represented by "xxx|xxxxx|xxxx||xxxx". The string representing the solution has a total of 20 symbols, 4 of which are vertical lines. The number of such strings is therefore $\binom{20}{4}$. (Which is 4845)