MATH 2600/STAT 2600, Theory of Interest
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Homework Sheet 4
Model Solutions

1. *John makes a deposit of \$2,000 at the end of every year into an account that pays interest at $j_{12} = 5\%$. How much is in the account at the end of 12 years?*

   $j_{12} = 5\%$ is equivalent to $j_1 = (1 + \frac{0.05}{12})^{12} - 1$. Therefore, after 12 years, his investment is worth $2000\frac{(1+\frac{0.05}{12})^{12\times 12}-1}{(1+\frac{0.05}{12})^{12}-1} = \$32,049.20$.

2. *Mr. Wilson wants to save up \$20,000 by the time his daughter goes to university in 9 years time. He plans to make monthly payments,starting in one month into an account that gives $j_4 = 2\%$ interest. What should the payment be each month if:*

   (a) The account uses simple interest for partial interest periods.

   He wants to accumulate \$20,000 after 36 quarters. To find the equivalent quarterly payments, we solve:

$$20000 = Q\left(\frac{1.005^{36} - 1}{0.005}\right)$$

$$Q = \frac{100}{1.005^{36} - 1} = 508.4387...$$

   To accumulate this at simple interest, we have $Q = M(1 + (1 + \frac{0.005}{3}) + (1 + \frac{0.005\times 2}{3})) = M(3.005)$. So the monthly payment is \$169.20.

   (b) The account uses compound interest for partial interest periods.

   Solution 1: We can use the same quarterly payment as in (a), but using simple interest we have $Q = M(1 + 1.005^{\frac{1}{3}} + 1.005^{\frac{2}{3}})$ so the monthly payment is \$169.20.

   Solution 2: $j_4 = 2\%$ is equivalent to $j_{12} = (1.005)^{\frac{1}{3}} - 1$, so we get

$$20000 = M\left(\frac{1.005^{36} - 1}{1.005^{\frac{1}{3}} - 1}\right)$$

$$M = 169.20$$

3. *Ms. King buys a house for \$250,000. She makes a down payment of \$50,000 and takes out a mortgage for the remaining \$200,000. The interest rate on the mortgage is $j_2 = 10\%$.*

   (a) If she takes a 20 year mortgage, what are the monthly payments, and what is the concluding smaller payment?

We have that $200000 = R\left(\frac{1-1.05^{-40}}{1.05^{\frac{1}{6}}-1}\right)$ $R = 1903.3288...$ So the monthly payment is \$1903.33. This monthly payment schedule gives a present value of $1903.33\left(\frac{1-1.05^{-40}}{1.05^{\frac{1}{6}}-1}\right) = 200000.120...$ After 20 years, the excess 12.... cents has a value of $0.120... \times 1.05^{40} = 0.85$, so the final payment is reduced by 85 cents, to give a final payment of \$1902.48.

(b) (i) If she can afford up to \$2500 a month, how many years should her mortgage be (it has to be a whole number of years)?

We want to find the value of n such that

$$200000 = 2500\left(\frac{1-1.05^{-2n}}{1.05^{\frac{1}{6}}-1}\right)$$

$$80(1.05^{\frac{1}{6}}-1) = 1-1.05^{-2n}$$

$$1.05^{-2n} = 1-80(1.05^{\frac{1}{6}}-1)$$

$$n = -\frac{\log(1-80(1.05^{\frac{1}{6}}-1)}{2\log(1.05)} = 10.8...$$

So it should be an 11 year mortgage.

(ii) What would the monthly payments and the final payment be in this case?

We have that $200000 = R\left(\frac{1-1.05^{-22}}{1.05^{\frac{1}{6}}-1}\right)$ $R = 2481.15...$ So the monthly payment is \$2481.16. This monthly payment schedule gives a present value of $2481.15\left(\frac{1-1.05^{-22}}{1.05^{\frac{1}{6}}-1}\right) = 200000.80...$

After 11 years the value of the excess would be $0.80... \times 1.05^{22} = 2.35$, so the final payment would be \$2478.81.

(c) (i) If she can invest her money at $j_{12} = 5\%$, how much would the money left over from her \$2500 a month, when following the payment schedule in (a), be worth at the end of 20 years?

Following the payment schedule in (a), she would have \$596.67 left over each month. Invested at $j_{12} = 5\%$, this would give $596.67\left(\frac{(1+\frac{0.05}{12})^{240}-1}{\frac{0.05}{12}}\right) =$ 245251.46 Also, with the reduction of 85 cents in the final payment, the total would be \$245,252.31.

(ii) What if she follows schedule (b)?

If she follows schedule (b), she will have \$18.84 left over each month for the first 11 years, and \$2500 left over each month for the remaining 9 years. Therefore, at $j_{12} = 5\%$, she will have $18.84\left(\frac{(1+\frac{0.05}{12})^{132}-1}{\frac{0.05}{12}}\right) = 3306.52...$ after 11 years. This will then accumulate to $3306.52... \times (1+\frac{0.05}{12})^{108} =$

5180.82... at the end of 20 years. Also, the reduction of \$2.35 on the last payment will accumulate to $2.35 \times (1+\frac{0.05}{12})^{108} = 3.68...$ at the end of 20 years. Finally, the \$2500 a month for the last 9 years will accumulate to $2500\left(\frac{(1+\frac{0.05}{12})^{108}-1}{\frac{0.05}{12}}\right) = 340107.99$, so the total value after 20 years would be \$345,291.49.

4. *How much money is needed to establish a scholarship fund that will pay out \$20,000 a year forever, if the money is invested at $j_4 = 6\%$?*

   $j_4 = 6\%$ is equivalent to $j_1 = 1.015^4 - 1$. Therefore, the amount of money needed is $\frac{20000}{1.015^4-1} = \$325,926.39$ (round up to ensure there is enough money).

5. *Andrew starts a savings account on 1st January, which pays $j_{12} = 4\%$ interest. He plans to make monthly deposits to the account for 25 years, starting on 1st January, until he retires. He starts by making a monthly deposit of \$200. Every year, on 1st January, his salary increases by 4%, and he therefore plans to increase his deposits by 4% at this time. If he keeps up this planned deposit schedule, how much will be in the account when he retires on 1st January in 25 years time (he doesn't make a deposit at this time)?*

   The deposits in the first year are equivalent to a single deposit of $200(1+\frac{0.04}{12})\frac{1-(1+\frac{0.04}{12})^{-12}}{\frac{0.04}{12}} = 2356.628...$ on 1st January. Similarly, the deposits n years later are equivalent to a deposit of $2356.628... \times 1.04^n$. Therefore, after 25 years, the accumulated value is

$$2356.628... \frac{((1+\frac{0.04}{12})^{12})^{25} - 1.04^{25}}{(1+\frac{0.04}{12})^{12} - 1.04} \times (1+\frac{0.04}{12})^{12} = \$158,523.80$$

6. *Joe has a salary of \$40,000 a year. Every year, he gets a 5% salary increase. He needs \$35,000 a year for his living expenses, and he saves the rest in an account which pays $j_1 = 3\%$ interest. How much will be in the account after he makes the 16th payment in 15 years time?*

   The amount invested after n years is $40000(1.05)^n - 35000$. We can calculate the accumulated value in the account as the difference between what would have been accumulated if he were to invest his entire salary and the equivalent value of his living expenses up to that point. If he were to invest his entire salary, the value would be: $40000\frac{1.05^{16}-1.03^{16}}{0.02} = \$1,156,336.29...$ On the other hand his living expenses would account for an accumulated value of $35000\frac{1.03^{16}-1}{0.03} = \$705,490.84...$ so the accumulated value of the money invested is $\$1,156,336.29... - \$705,490.84... = \$450,845.45$