MATH 2600/STAT 2600, Theory of Interest
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Homework Sheet 7
Model Solutions

1. *Calculate the modified duration and Macauley duration of a 10-year bond with semi-annual coupons at coupon rate 12%, if it is purchased for a yield of:*

   (a) $j_2 = 2\%$.

   The duration is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.01}{0.01} - \frac{1.01+20(0.05)}{0.06(1.01^{20}-1)+0.01} =$ 14.40 years.

   (b) $j_2 = 12\%$.

   The duration is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.06}{0.06} - \frac{1.06}{0.06(1.06^{20}-1)+0.06} =$ 12.16 years.

   (c) $j_2 = 22\%$.

   The duration is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.11}{0.11} - \frac{1.11-20(0.05)}{0.06(1.11^{20}-1)+0.11} =$ 9.88 years.

2. *A company expects to receive \$2,000,000 in 2 years time, and pay out \$5,000,000 in 5 years time. If the current spot rates are as in the following table:*

Term(years)	2	5	7	8
rate	*3.5%*	*4.4%*	*5%*	*5%*

   (a) find a way for the company Reddington immunise these cash-flows by buying zero-coupon bonds with maturities in 7 or 8 years.

   The present value of these payments is $2000000(1.035)^{-2} - 5000000(1.044)^{-5} = -2164486.46$. The modified duration is $\frac{2\times2000000(1.035)^{-3}-5\times5000000(1.044)^{-6}}{-2164486.46} = 7.253$. 7-year strip bonds have a modified duration of $\frac{7}{1.05} = 6.6667$, and 8-year strip bonds have a modified duration of $\frac{8}{1.05} = 7.6190$. To immunise the payments, the company needs to buy bonds with modified duration 7.253. If it buys \$1 of 7-year bonds, and x of 8-year bonds, then the modified duration of this portfolio is $\frac{6.6667+7.6190x}{1+x} = 7.253$. We rewrite this as

$$7.6190 + \frac{6.6667 - 7.6190}{1+x} = 7.253$$

$$x = \frac{6.6667 - 7.6190}{7.253 - 7.6190} - 1 = \frac{6.6667 - 7.253}{7.253 - 7.6190} = 1.6057$$

If the company buys \$1 of the 7-year bonds, and \$1.6057 of the 8-year bonds, the present value will be $(1.05)^{-7} + 1.6057(1.05)^{-8} = 1.7975$. To immunise its liabilities, it needs the present value to be \$2,164,486.46, so it needs to buy \$1204162.73 of 7-year bonds, and \$1,933,561.44 of 8-year bonds.

(b) Is the immunisation in (a) a full immunisation?

3. *Find two payments in 5 and 16 years that immunise the liabilities of a 20-year bond with face value \$80,000 and semi-annual coupon rate 4% at a price to yield 4.6%.*

   The Macauley duration of the bond is given by $D = \frac{1.023}{0.023} - \frac{1.023-40\times 0.003}{0.02(1.023^{40}-1)+0.023} =$ 27.33 coupon periods, or 13.666 years. Therefore, the modified duration is $\frac{27.33}{1.023} = 26.718$, so a set of payments which immunises the bond must also have modified duration 26.718. A payment in 5 years time has Macauley duration 5, so has modified duration $10 \times 1.023 = 10.23$, and a payment in 16 years has modified duration $32 \times 1.023 = 32.736$. To achieve a modified duration of 26.718, we need the present value of the 16-year bond to be $\frac{26.718-10.23}{32.736-10.23} = 0.7326$ of the total present value. The total present value must be the present value of the bond, which is given by Makeham's formula $K = 80000(1.023)^{-20} = 32215.48$, so $P = 32215.48 + (80000 - 32215.48)\frac{2}{2.3} = 73767.24$, so the present value of the payment after 16 years is $73767.24 \times 0.7326 = 54041.63$, and the value after 16 years is . $54041.63(1.023)^{32} = 111879.05$. The present value of the payment after 5 years is $73767.24 - 54041.63 = 19725.60$, and the value after 5 years is $19725.60(1.023)^{10} = 24762.05$.

4. *The current term structure has the following yields on zero-coupon bonds:*

Term(years)	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
rate	*4%*	*4.2%*	*4.7%*	*5%*	*5.3%*	*5.6%*	*5.8%*	*6%*

   Calculate the modified duration of a 10% semi-annual 4-year bond, based on a parallel shift in the term structure.

   The present value of the bond with face value \$100 is $5(1.02)^{-1}+5(1.021)^{-2}+5(1.0235)^{-3}+5(1.025)^{-4}+5(1.0265)^{-5}+5(1.028)^{-6}+5(1.029)^{-7}+105(1.03)^{-8} = 114.50$

   The rate of change of the present value with respect to a parallel shift in interest rates is $-1\times 5\times 1.02^{-2} - 2\times 5\times 1.021^{-3} - 3\times 5\times 1.0235^{-4} - 4\times 5\times 1.025^{-5} - 5\times 5\times 1.0265^{-6} - 6\times 5\times 1.028^{-7} - 7\times 5\times 1.029^{-8} - 8\times 5\times 1.03^{-9} = -763.28$. The modified duration is therefore $\frac{763.28}{114.50} = 6.67$.