MATH 2600/STAT 2600, Theory of Interest
FALL 2014
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Practice Final Examination
Model Solutions

1. *Calculate the price that should be paid for each of the following bonds to obtain the desired yield:*

   (a) Face value \$70,000, maturing at par in 12 years, coupon rate $j_2 = 7\%$, desired yield $j_2 = 4\%$. [5 mins]

   We use Makeham's formula:

   $K = 70000(1.02)^{-24} = 43520,50$. $P = 43520.50 + (70000 - 43.52050)\frac{7}{4} =$ \$89, 859.62

   (b) Face value \$180,000, maturing at par in 38 years, coupon rate $j_2 = 2\%$, desired yield $j_2 = 4.5\%$. [5 mins]

   We use Makeham's formula:

   $K = 180000(1.0225)^{-76} = 33178.78$. $P = 33178.78+(180000-33178.78)\frac{2}{4.5} =$ \$98, 432.66.

2. *(a) Write out a complete bond amortisation schedule for a bond with face value \$20,000 with coupon rate $j_2 = 3\%$, maturing at par in 2 years, sold to an investor who wishes to receive a yield of $j_2 = 5\%$ [10 mins]*

   We use Makeham's formula:

   $K = 20000(1.025)^{-4} = 18119.01$. $P = 18119.01 + (20000 - 18119.01)\frac{3}{5} =$ \$19, 247.61. This gives the schedule:

	Outstanding Balance	Coupon	Interest	Principal Repaid
1	\$19, 247.61	\$300	\$481.19	\$ − 181.19
2	\$19, 428.80	\$300	\$485.72	\$ − 185.72
3	\$19, 614.52	\$300	\$490.36	\$ − 190.36
4	\$19, 804.88	\$20300	\$495.12	\$19, 804.88

   (b) Write out a complete bond amortisation schedule for a bond with face value \$15,000 with coupon rate $j_2 = 5\%$, maturing at par in 2.5 years, sold to an investor who wishes to receive a yield of $j_2 = 3\%$ [10 mins]

   We use Makeham's formula:

   $K = 20000(1.025)^{-4} = 13923.90$. $P = 13923.90 + (20000 - 13923.90)\frac{3}{5} =$ \$15, 717.40. This gives the schedule:

	Outstanding Balance	Coupon	Interest	Principal Repaid
1	\$15, 717.40	\$375	\$235.76	\$139.24
2	\$15, 578.16	\$375	\$233.67	\$141.33
3	\$15, 436.83	\$375	\$231.55	\$143.45
4	\$15, 293.38	\$375	\$229.40	\$145.60
5	\$15, 147.78	\$15, 375	\$227.22	\$15, 147.78

3. *A bond has face value \$30,000, maturity in 20 years, coupon rate $j_2 = 6\%$. After 2 years and 4 months, it is sold to Mr. Armstrong, who wishes to receive a yield of $j_2 = 4\%$. Calculate*

   (a) The flat price. [5 mins]

   After 2 years, the bond has 18 years to maturity, so we can use Makeham's formula, $K = 30000(1.02)^{-36} = 14706.69$, and $P = 14706.69 + (30000 - 14706.69)\frac{6}{4} = 37646.65$. We obtain the flat price by multiplying by $(1.02)^{\frac{4}{6}}$ to get \$38,146.95

   (b) The quoted price. [5 mins]

   The accrued interest is $900 \times \frac{4}{6} = 600$, so the quoted price is \$37546.95.

4. *Mr. Bruce buys a zero-coupon bond with face value \$37,000, maturing at par in 9 years, for a price to yield 3.7% annually. After three years, interest rates increase, and he sells the bond to an investor who wishes to receive a yield of 4.6% annually. What is Mr. Bruce's rate of return? [5 mins]*

   The price Mr. Bruce pays is $37000(1.037)^{-9} = 26680.44$. The price the investor pays three years later is $37000(1.047)^{-6} = 28249.56$, so Mr. Bruce achieves a return of $\frac{28249.56}{26680.44} - 1 = 5.88\%$ in three years, so his annual rate of return is $\left(\frac{28249.56}{26680.44}\right)^{\frac{1}{3}} - 1 = 1.92\%$.

5. *A company are considering a project. The project has the following expected cash flows (all amounts are calculated at the beginning of the year):*

Year	*0*	*1*	*2*	*3*	*4*	*5*
Net Cash Flow (000)	*-100*	*10*	*20*	*20*	*10*	*50*

   (a) What is the Net Present value of the project at $j_1 = 2\%$? [5 mins]

   The Net Present value at $j_1 = 2\%$ is $50(1.02)^{-5}+10(1.02)^{-4}+20(1.02)^{-3}+20(1.02)^{-2} + 10(1.02)^{-1} - 100 = 2.39874$.

   (b) What is the Net Present value of the project at $j_1 = 4\%$? [5 mins]

   The Net Present value at $j_1 = 2\%$ is $50(1.04)^{-5}+10(1.04)^{-4}+20(1.04)^{-3}+20(1.04)^{-2} + 10(1.04)^{-1} - 100 = -4.46917$.

   (c) Which of the following is the internal rate of return? (Justify your answer.) [10 mins]

   (i) 2.54% (ii) 2.68% (iii) 2.72% (iv) 5.10%

   We calculate the Net present value for each case:

   (i) $50(1.0254)^{-5}+10(1.0254)^{-4}+20(1.0254)^{-3}+20(1.0254)^{-2}+10(1.0254)^{-1}-100 = 0.47596$.

   (ii) $50(1.0268)^{-5}+10(1.0268)^{-4}+20(1.0268)^{-3}+20(1.0268)^{-2}+10(1.0268)^{-1}-100 = -0.01405$.

(iii) $50(1.0272)^{-5}+10(1.0272)^{-4}+20(1.0272)^{-3}+20(1.0272)^{-2}+10(1.0272)^{-1}-100=-0.15342$.

(iv) $50(1.051)^{-5}+10(1.051)^{-4}+20(1.051)^{-3}+20(1.051)^{-2}+10(1.051)^{-1}-100=-7.96564$.

so **(ii)** is the internal rate of return.

6. *A company are considering two projects. The projects have the following expected cash flows (all amounts are calculated at the beginning of the year):*

Year	*0*	*1*	*2*	*3*	*4*
Project 1 Net Cash Flow (000)	*-150*	*20*	*40*	*70*	*90*
Project 2 Net Cash Flow (000)	*-110*	*40*	*40*	*40*	*40*

(a) Which project should they prefer if the cost of capital is $j_1 = 2\%$*? [10 mins]*

At $j_1 = 2\%$, the net present value of Project 1 is $90(1.02)^{-4}+70(1.02)^{-3}+40(1.02)^{-2}+20(1.02)^{-1}-150 = 57.16324$. The net present value of Project 2 is $40(1.02)^{-4}+40(1.02)^{-3}+40(1.02)^{-2}+40(1.02)^{-1}-110 = 42.30915$, so they should prefer Project 1.

(b) Which project should they prefer if the cost of capital is $j_1 = 12\%$*? [10 mins]*

At $j_1 = 12\%$, the net present value of Project 1 is $90(1.12)^{-4}+70(1.12)^{-3}+40(1.12)^{-2}+20(1.12)^{-1}-150 = 6.76614$. The net present value of Project 2 is $40(1.12)^{-4}+40(1.12)^{-3}+40(1.12)^{-2}+40(1.12)^{-1}-110 = 11.49397$, so they should prefer Project 2.

7. *Mrs. Chapman has \$300,000 in her investment fund at the start of the year. One month later, she withdraws \$140,000. Another 3 months later (4 months from the start of the year), she withdraws a further \$90,000 (there is enough money in the fund to cover this withdrawl). After another 5 months, (9 months from the start of the year), she invests \$160,000 in the fund. At the end of the year, she has \$255,000 in her account. What is her dollar-weighted rate of return for the year? [10 mins]*

This is an equation of value with focal date the end of the year at simple interest, so we get $255000 = 300000(1+i)-140000\left(1+i\frac{11}{12}\right)-90000\left(1+i\frac{8}{12}\right)+160000\left(1+i\frac{3}{12}\right)$. This gives $\left(300000-\frac{11}{12}140000-\frac{8}{12}90000+\frac{3}{12}160000\right)i = 45000$, so $i = 16.48\%$.

8. *Mr. Davies is managing an investment fund. At the start of the year, there is \$8,000,000 in the fund. The fund activity is summarised in the following table:*

Months from start of year	*Fund value before transaction*	*Net Deposit*	*Fund value after transaction*
0			*8,000,000*
1	*7,920,000*	*-1,000,000*	*6,920,000*
3	*7,150,000*	*500,000*	*7,650,000*
4	*7,720,000*	*100,000*	*7,820,000*
5	*7,830,000*	*600,000*	*8,430,000*
6	*8,660,000*	*-400,000*	*8,260,000*
8	*7,840,000*	*500,000*	*8,340,000*
10	*8,220,000*	*-100,000*	*8,120,000*
12	*8,400,000*		*8,400,000*

Calculate his time-weighted rate of return for the year. [10 mins]

His time-weighted rate of return is

$$\left(\frac{792}{800}\right)\left(\frac{715}{692}\right)\left(\frac{772}{765}\right)\left(\frac{783}{782}\right)\left(\frac{866}{843}\right)\left(\frac{784}{826}\right)\left(\frac{822}{834}\right)\left(\frac{840}{812}\right)-1 = 2.75\%$$

9. *Mr. Collins is managing an investment fund. At the start of the year, there is \$8,000,000 in the fund. The fund activity is summarised in the following table:*

Months from start of year	*Fund value before transaction*	*Net Deposit*	*Fund value after transaction*
0			*8,000,000*
3	*8,150,000*	*500,000*	*8,650,000*
5	*8,530,000*	*600,000*	*9,130,000*
10	*8,820,000*	*-400,000*	*8,420,000*
12	*8,600,000*		*8,600,000*

Calculate

(a) The time-weighted rate of return for the year.

The time-weighted rate of return is given by $\left(\frac{815}{800}\right)\left(\frac{853}{865}\right)\left(\frac{882}{913}\right)\left(\frac{860}{842}\right) - 1 = -0.87\%$.

(b) The dollar-weighted rate of return for the year.

The dollar-weighted rate of return is given by solving the equation of value

$$8000000(1+i)+500000(1+\frac{9}{12}i)+600000(1+\frac{7}{12}i) = 400000(1+\frac{2}{12}i)+860000$$

This gives

$$(8000000 + 375000 + 350000 - 66666.66667)i = -100000$$

which gives

$$i = -\frac{100000}{8658333.33} = -1.15\%$$

10. *A company has an account which pays interest at $j_1 = 3\%$ on credit balances, and charges interest at $j_1 = 5\%$ on debit balances. The company is considering using this account to fund possible projects with the following expected cashflows: (All amounts are calculated at the begining of the year.)*

Year	*0*	*1*	*2*	*3*	*4*	*5*
Net Cash Flow (000)	*-150*	*30*	*50*	*90*	*50*	*20*

How much money does the company have in its account at the start of the fifth year, when it receives the final cash-flow, if it invests in this project? [10 mins]

If it invests in the project, it has a debit balance until the start of the third year, when its balance is $90+50(1.05)+30(1.05)^2-150(1.05)^3 = 1.93125$. From this point on, the balance grows with an interest rate of 3%, so the balance after receiving the final payment is $1.93125(1.03)^2+50(1.03)+20 = 73.54886$, so the company has $73,548.86.

11. *The current term structure has the following semi-annual yields on zero-coupon bonds:*

Term(years)	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
rate	*4.7%*	*4.9%*	*5.2%*	*5.3%*	*5.3%*	*5.4%*	*5.4%*	*5.3%*

How much should be paid for a $100 face-value bond with semi-annual coupons, maturing at par in 4 years time, with 6% annual coupon rate?

We calculate this as the sum of the present values of the coupons:

$$3(1.0235)^{-1}+3(1.0245)^{-2}+3(1.026)^{-3}+3(1.0265)^{-4}+3(1.0265)^{-5}+3(1.027)^{-6}+3(1.027)^{-7}+103(1.0265)^{-}$$

12. *The current term structure has the following semi-annual yields on zero-coupon bonds:*

Term(years)	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
rate	*6%*	*6.2%*	*6.2%*	*5.8%*	*5.3%*	*5.1%*	*4.8%*	*4.7%*

How much should be paid for a $100 face-value bond with semi-annual coupons, maturing at par in 4 years time, with 6% annual coupon rate? [10 mins]

This is just the sum of the present values of all coupons and the final redemption value. That is

$$3(1.03)^{-1}+3(1.031)^{-2}+3(1.031)^{-3}+3(1.029)^{-4}+3(1.0265)^{-5}+3(1.0255)^{-6}$$
$$+3(1.024)-7+103(1.0235)^{-8} = 104.43$$

13. *For the term structure in Q. 12, what is the implied forward rate for an 18-month loan starting in 1 year's time? [5 mins]*

    The implied forward rate over a 6-month period is given by $\left(\frac{(1.0265)^5}{(1.031)^2}\right)^{\frac{1}{3}} - 1 = 2.351\%$, so the implied forward rate is $j_2 = 4.70\%$. [The annual effective rate is 4.76%.]

14. *The spot-rates for 1, 2 and 3 year strip bonds are 4%, 4.3% and 4.2% annually. You have the opportunity to borrow or lend money at these rates, and you also have the opportunity to arrange to borrow money in two year's time for one year, at an annual rate of 4.4%, or to lend money in two years time for one year at an annual rate of 4.2%. Can you construct an arbitrage possibility? [8 mins]*

    The implied forward rate for a one-year loan, starting in two year's time is $\frac{(1.042)^3}{(1.043)^2} - 1 = 4.00\%$, which is lower than the rate at which you can lend money in a forward rate agreement, so you can construct an arbitrage as follows. Arrange to lend \$1,000,000 in two years' time for one year at 4.2%, borrow $1000000(1.043)^{-2}$ or \$919,245.26 for three years at 4.2%, and invest it for two years at 4.3%.

    After two years, this \$919,245.26 has grown to \$1,000,000 which you then lend at 4.2% for a year. After this year, you recieve \$1,042,000 and you owe $919245.26(1.042)^3$ or \$1,040,002.88 from the original loan. This leaves you a profit of \$1,997.12.

15. *The spot-rates for 2 and 3 year strip bonds are 3.4% and 3.5% annually. You have the opportunity to borrow or lend money at these rates, and you also have the opportunity to arrange to borrow money in two year's time for one year, at an annual rate of 3.9%, or to lend money in two years time for one year at an annual rate of 3.8%. Can you construct an arbitrage possibility?*

    The implied forward rate for a one-year loan starting in two years time is $\frac{(1.035)^3}{(1.034)^2} - 1 = 3.7\%$. Since this is lower than the rate at which you can lend money, it is possible to construct an arbitrage as follows:

    - Arrange to lend \$1,000,000 in two years time, at 3.8% for one year.
    - Invest $1000000(1.034)^{-2} = 935317.20$ at 3.4% for two years.
    - Borrow \$935,317.20 at 3.5% for three years.

16. *Mr and Mrs. Eastman are borrowing \$70,000 at a variable rate of prime+2.4%. They are making interest-only payments annually. They make a forward rate agreement with the bank, so that the interest rate for the third year (starting two years from now) will be 6%. In two years time, the prime rate is 4%. How much money changes hands at the end of the third year, and in which direction? [10 mins]*

The payment due on the original loan is 6.4% of $70,000, which is $4,480. The payment agreed in the forward rate agreement is 6% of $70,000, which is $4,200, so the bank pays the difference, which is $280 to Mr. and Mrs. Eastman.

17. *Mrs. Funk can borrow at 8% on the fixed-rate market, or at prime+6.5% on the variable rate market. Dr. Green can borrow at 5% on the fixed-rate market, or at prime+2.9% on the variable rate market. Mr. Harris offers to arrange interest rate swaps with both of them so that Mrs. Funk can borrow $200,000 at prime+6.3% and Dr. Green can borrow $200,000 at 4.8%. How much spread income does Mr. Harris make on this transaction? [10 mins]*

    If Mrs. Funk and Dr. Green both borrow $200,000 on appropriate markets (Mrs. Funk borrowing on the fixed-rate market, and Dr. Green borrowing on the variable rate market) the total interest would be prime+2.9%+8%, or prime+10.9% of $200,000. Under the arrangements with Mr. Harris, the total interest they are paying is prime+6.3%+4.8%, or prime+11.1% of $200,000. Mr. Harris' spread income is the difference between this and the total interest they could achieve on their own, or 0.02% of $200,000, which is $400 a year.

18. *Mr. Dunn can borrow at 4% on the fixed-rate market, or at prime+1.5% on the variable rate market. Mr. Evans can borrow at 5% on the fixed-rate market, or at prime+2% on the variable rate market. Mrs. Foley offers to arrange interest rate swaps with both of them so that Mr. Dunn can borrow $300,000 at prime+1.3% and Mr. Evans can borrow $300,000 at 4.8%. How much spread income does Mrs. Foley make on this transaction?*

    Mr. Dunn starts by borrowing at 4% and Mr. Evans starts by borrowing at prime+2%, so the total repayment to the bank is prime+6%, but with the swaps, Mr. Dunn's net cost is prime+1.3% and Mr. Evans' net cost is 4.8%, so the total net cost is prime+6.1%, so the difference between the net cost and the repayment to the bank is Mrs. Foley's spread income, that is 0.1% of $300,000, or $300.

19. *The current term structure has the following annual yields on zero-coupon bonds:*

Term(years)	1	2	3	*4*	5
rate	*3%*	*3.1%*	*3.2%*	*3.4%*	*3.4%*

    Mrs. Grey has a floating rate loan of $800,000, with annual interest-only payments. She wishes to exchange this for a fixed rate over the next 5 years (i.e. she wants to pay the same interest rate over the next 5 years). What should this rate be?

    The present value of the interest payments due over the next 5 years is the difference between the present value of the loan and the present value of the outstanding balance after 5 years. That is, $800000 - 800000(1.034)^{-5} = 123158.01$. We solve for the annual payment X over the next 5 years that has this present value:

$$X((1.03)^{-1} + (1.031)^{-2} + (1.032)^{-3} + (1.034)^{-4} + (1.034)^{-5}) = 123158.01$$

$$X = \frac{123158.01}{(1.03)^{-1} + (1.031)^{-2} + (1.032)^{-3} + (1.0}$$

$$= 27113.32$$

This gives a yearly rate of $\frac{27113.32}{800000} = 3.39\%$.

20. *The current term structure has the following annual yields on zero-coupon bonds:*

Term(years)	1	2	3	*4*	5	6
rate	*4.2%*	*4.1%*	*4.3%*	*4.5%*	*4.6%*	*4.8%*

Mr and Mrs. Ive have a floating rate loan of \$600,000, with annual interest-only payments. They wishes to exchange this for a fixed rate over the next 6 years (i.e. They want to pay the same interest rate over the next 6 years). What should this rate be? [10 mins]

The present value of the interest payments made can be found by considering the outstanding balance after 6 years, which is still \$600,000. The present value of this is $600000(1.048)^{-6} = 452880.43$. The present value of the payments made must therefore be $600000 - 452880.43 = 147119.57$. If the equal payments are X, then we have the equation of value

$$147119.57 = X(1.042)^{-1} + X(1.041)^{-2} + X(1.043)^{-3} + X(1.045)^{-4} + X(1.046)^{-5} + X(1.048)^{-6}$$

This gives $X = \frac{147119.57}{(1.042)^{-1}+(1.041)^{-2}+(1.043)^{-3}+(1.045)^{-4}+(1.046)^{-5}+(1.048)^{-6}} = 28534.74$.

21. *For the spot rates from Q. 20, which of the following is the at-par yield of a 6-year bond with annual coupons? [10 mins]*

(i) 4.60% (ii) 4.68% (iii) 4.72% (iv) 4.76%

The prices for a bond with face value \$100 at the given coupon rates are:

(i) $4.6(1.042)^{-1} + 4.6(1.041)^{-2} + 4.6(1.043)^{-3} + 4.6(1.045)^{-4} + 4.6(1.046)^{-5} + 104.6(1.048)^{-6} = 99.20$

(ii) $4.68(1.042)^{-1} + 4.68(1.041)^{-2} + 4.68(1.043)^{-3} + 4.68(1.045)^{-4} + 4.68(1.046)^{-5} + 104.68(1.048)^{-6} = 99.61$

(iii) $4.72(1.042)^{-1} + 4.72(1.041)^{-2} + 4.72(1.043)^{-3} + 4.72(1.045)^{-4} + 4.72(1.046)^{-5} + 104.72(1.048)^{-6} = 99.82$

(iv) $4.76(1.042)^{-1} + 4.76(1.041)^{-2} + 4.76(1.043)^{-3} + 4.76(1.045)^{-4} + 4.76(1.046)^{-5} + 104.76(1.048)^{-6} = 100.02$

so the at-par yield is 4.76%.

22. *Calculate the modified duration and Macauley duration of a 10-year bond with semi-annual coupons at coupon rate 8%, if it is purchased for a yield of 6%. [10 mins]*

The Macauley duration is given by

$$D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i}$$

Which in this case gives

$$D = \frac{1.03}{0.03} - \frac{1.03+20(0.01)}{0.04(1.03^{20}-1)+0.03} = 14.57$$

coupon periods, or 7.29 years. The modified duration is given by $\frac{D}{1+i} = \frac{14.57}{1.03} = 14.15$. This is the sensitivity to change in the rate for half a year (or $\frac{j_2}{2}$) so the sensitivity to change in j_2 is a modified duration of 7.07. [For a change in effective rate it is 6.87.]

23. *A company expects to pay \$2,000,000 in 4 years time, and \$4,000,000 in 6 years time. The current annual effective interest rate is 5%. Find a way for the company to Reddington immunise these cash-flows by buying zero-coupon bonds with maturities in 2 or 9 years. [15 mins]*

The total present value of the companies liabilities is $2000000(1.05)^{-4} + 4000000(1.05)^{-6} = 4630266.54$.

The duration is given by $\frac{4\times 2000000(1.05)^{-4}+6\times 4000000(1.046)^{-6}}{4630266.54} = 5.29$ years.

If the company arranges assets of A_2 after 2 years and A_9 after 9 years, the present value of these assets will be $A_2(1.05)^{-2}+A_9(1.05)^{-9} = 4630266.54$ and the duration will be $\frac{2A_2(1.05)^{-2}+9A_9(1.05)^{-9}}{4630266.54} = 5.29$. which gives $2A_2(1.05)^{-2}+9A_9(1.05)^{-9} = 5.29 \times 4630266.54 = 24490789.32$. Subtracting 2 times the first equation gives

$$(1.05)^{-9}(9-2)A_9 = 24490789.32 - 2\times 4630266.54 = 15230256.25$$

so

$$A_9 = (1.05)^9\frac{15230256.25}{7} = \$3,375,303.75$$

The present value of A_9 is just $\frac{15230256.25}{7} = 2175750.89$, so the present value of A_2 must be $4630266.54 - 2175750.89 = 2454515.64$, so we must have $A_2 = 2454515.64(1.05)^2 = \$2,706,103.50$.

Alternatively:

The duration of A_2 is 2 and the duration of A_9 is 9, so if the present values of A_2 and A_9 are proportional to a and b respectively, with $a+b = 1$, then the duration of both payments is $2a + 9(1-a) = 5.29$, which gives $a = \frac{9-5.29}{9-2} = 0.5301024519$. We know that the total present value of A_2 and A_9 is \$4,630,266.54 and A_2 makes up 53.01% of that, so the present values of A_2 and A_9 are $0.5301 \times 4630266.54 = 2454515.64$ and $(1 - 0.5301) \times 4630266.54 = 2175750.89$ respectively. This gives $A_2 = \$2,706,103.50$. and $A_9 = \$3,375,303.75$.

24. *A company needs to pay \$3,000,000 in 5 years time, and \$3,000,000 in 6 years time.*

    (a) If the current interest rate is $j_1 = 5\%$, find a way for the company to Redington immunise these cash-flows by buying zero-coupon bonds with maturities in 3 or 8 years.

    The present values of the liabilities are $3000000(1.05)^{-5} = 2350578.50$ and $3000000(1.05)^{-6} = 2238646.19$, so the total present value of the liabilities is 4589224.69. The duration of the liabilities is $\frac{5\times2350578.50+6\times2238646.19}{4589224.69} = 5.4878$ years. We need to arrange the duration of the assets to be 5.4878 years, and the present value of the assets to be 4589224.69. That is, we need to solve $PV(A_3) + PV(A_8) = 4589224.69$ and $\frac{3PV(A_3)+8PV(A_8)}{4589224.69} = 5.4878$. This gives $5PV(A_8) = 5.4878 \times 4589224.69 - 3 \times 4589224.69$, so $PV(A_8) = 2283419.11$, and so $PV(A_3) = 2305805.58$. This gives $A_3 = 2305805.58(1.05)^3 = 2669258.18$ and $A_8 = 2283419.11(1.05)^8 = 3373650.00$.

    (b) One year later, the interest rate has dropped to $j_1 = 4\%$. The company sells some of the bonds and buys some other bonds to retain an immunised portfolio. What is the total value of bonds sold? [The company only sells and buys bonds with the same maturity as in its origninal portfolio.]

    At an interest rate $j_1 = 4\%$, the value of the assets is $A_3(1.04)^{-2} + A_8(1.04)^{-7} = 2669258.18(1.04)^{-2} + 3373650.00(1.04)^{-7} = 5031575.97$. The value of the liabilities is $3000000(1.04)^{-4}+3000000(1.04)^{-5} = 5030193.89$. The company sells bonds to make the present values matched again. This means that the total value of bonds sold must be the difference in the present values, namely $5031575.97 - 5030193.89 = 1382.07$. [Note that since this is one year later, all discount factors correspond to a time period one year less than in part (a).]

    [The new duration of the liabilities is $\frac{4\times3000000(1.04)^{-4}+5\times3000000(1.04)^{-5}}{5030193.89} = 4.49$ years. The company therefore wants the proportion of 2-year bonds to be $a = \frac{7-4.49}{7-2} = .5020$, so the new present value of 2-year bonds should be $5030193.89 \times 0.5020 = \$2,524,960.07$ and the present value of 7-year bonds should be \$2,505,233.82. The present value of 2-year bonds it holds is $2669258.18(1.04)^{-2} = 2467879.23$, so it needs to buy an additional \$57,080.84 of 2-year bonds. The value of 7-year bonds it needs to hold is \$2,505,233.82, but it currently holds $3373650.00(1.04)^{-7} = \$2,563,696.73$, so it can sell \$58,462.91 of the 7-year bonds, for an net profit of \$1382.07.]

25. *Explain the difference between full immunisation and Reddington immunisation. [5 mins]*

Under Redington immunisation the present values of assets and liabilities are matched, as are their first derivatives with respect to interest rate, while the second derivatives with respect to interest rate of the present value of assets must be greater than the second derivatives with respect to interest rate of the present value of liabilities. This means that for sufficiently small changes in interest rates, the value of assets will always exceed that of liabilities. However, for larger changes in interest rates, it may be possible for the present value of liabilities to exceed that of assets.

Under full immunisation, whatever the interest rate is, the present value of assets is certain to be at least as large as the value of liabilities.

26. *Find two payments in 5 and 16 years that immunise the liabilities of a 15-year bond with face value \$120,000 and semi-annual coupon rate 7% at a price to yield $j_2 = 5\%$. [15 mins]*

 The Macauley duration of the bond is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.025}{0.025} - \frac{1.025+30(0.01)}{0.035(1.025^{30}-1)+0.025} = 20.106$ coupon periods, or 10.05 years.

 To match the durations, we need the present values of A_5 and A_{16} to be in proportions a and $1-a$ so that $5a + 16(1-a) = 10.05$, which gives $a = \frac{16-10.05}{16-5} = 0.5406$.

 The present value of the bond is calculated using Makeham's formula:

 $K = 120000(1.025)^{-30} = 57209.12$, and $P = 57209.12+(120000-57209.12)\frac{7}{5} = 145116.35$. Therefore the present values of A_5 and A_{16} are $0.5406 \times 145116.35 = 78456.20$ and $0.4594 \times 145116.35 = 66660.15$.

 Therefore, $A_5 = 78456.20(1.025)^{10} = 100430.56$ and $A_{16} = 66660.15(1.025)^{32} = 146902.78$.

27. *Calculate the modified duration and Macauley duration of a 10-year bond with semi-annual coupons at coupon rate 8%, if it is purchased for a yield of 5%.*

 The Macauley duration is given by the formula

 $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.025}{0.025} - \frac{1.025+20(0.015)}{0.04((1.025)^{20}-1)+0.025} = 14.7856$ coupon periods, or 7.3928 years.

 The modified duration is then given by dividing the Macauley duration by $1+i$ or 1.025 to get $\text{MD} = \frac{7.3928}{1.025} = 7.2124$.

28. *A 20-year semiannual bond has a current price of \$94.52, and a duration of 16.43 years. The current nominal interest rate is 3.24%. The next day, the relevant nominal interest rate increases to 3.26%. What is the approximate price of the bond? [You may ignore the one day's interest accrued, and only use the linear term in the interest rate.]*

 The duration of 16.43 years corresponds to a 32.86 coupon periods. The modified duration with respect to changes in the rate per period is therefore $\frac{32.86}{1.0162} = 32.336$. The rate per period has increased by 0.01%, so the price of the bond will decrease by $32.336 \times 0.01\% = 0.32336\%$, so the new price is $94.52 \times 0.9967664 = \$94.21$.

Alternative approximation:

Since the bond has a duration of 16.43 years, it behaves like a zero-coupon bond with maturity in 16.43 years. The face value of this zero coupon bond is $94.52(1.0162)^{32.86}$, so the new present value is $94.52(1.0162)^{32.86}(1.0163)^{-32.86} = \$94.21$. [Note that this is a different non-linear way of approximating the bond price, so could yield a slightly different answer from the approach above.]

29. *The current term structure has the following yields on zero-coupon bonds:*

Term(years)	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
rate	*6.3%*	*6.5%*	*7%*	*7.1%*	*7.1%*	*7.2%*

Calculate the modified duration of an 8% semi-annual 3-year bond, based on a parallel shift in the term structure. [15 mins]

The present value of the bond with face value 100 is

$$4(1.0315)^{-1}+4(1.0325)^{-2}+4(1.035)^{-3}+4(1.0355)^{-4}+4(1.0355)^{-5}+104(1.036)^{-6} = 102.19$$

The derivative of the present value with respect to a parallel shift in the term structure is

$$-(4(1.0315)^{-2} + 2 \times 4(1.0325)^{-3} + 3 \times 4(1.035)^{-4} + 4 \times 4(1.0355)^{-5} + 5 \times 4(1.0355)^{-6} \\ +6 \times 104(1.036)^{-7}) = 542.36$$

The modified duration is therefore $\frac{542.36}{102.19} = 5.307$. This is with respect to a change in the rate every half-year. To get the modified duration with respect to j_2, we need to divide this by 2, to get 2.15

Formulae

Macauley duration of a coupon bond:

$$D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r\left((1+i)^n - 1\right) + i}$$

Increasing annuities:

$$(Ia)_{\overline{n}|i} = \frac{(1+i)^{-1}a_{\overline{n}|i} - n(1+i)^{-n}}{i}$$