# MATH 2600/STAT 2600, Theory of Interest
# FALL 2014
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### Homework Sheet 7
### Model Solutions

1. *Calculate the modified duration and Macauley duration of a 12-year bond with semi-annual coupons at coupon rate 8%, if it is purchased for a yield of:*

   *(a)* $j_2 = 2\%$.

   The Macauley duration is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.01}{0.01} - \frac{1.01+24(0.03)}{0.04((1.01)^{24}-1)+0.01} = 17.78$ periods, or 8.84 years. The modified duration is $17.78(1.01)^{-1} = 17.61$ with respect to changes in the rate for a half-year period, or 8.80 with respect to changes in $j_2$.

   *(b)* $j_2 = 12\%$.

   The Macauley duration is given by $D = \frac{1+i}{i} + \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.06}{0.06} - \frac{1.06-24(0.02)}{0.04((1.06)^{24}-1)+0.06} = 14.48$ periods, or 7.24 years. The modified duration is $14.48(1.06)^{-1} = 13.66$ with respect to changes in the rate for a half-year period, or 6.83 with respect to changes in $j_2$.

   *(c)* $j_2 = 22\%$.

   The Macauley duration is given by $D = \frac{1+i}{i} + \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.11}{0.11} - \frac{1.11-24(0.07)}{0.04((1.11)^{24}-1)+0.11} = 11.1096$ periods, or 5.55 years. The modified duration is $11.11(1.06) = 10.009$ with respect to changes in the rate for a half-year period, or 5.00 with respect to changes in $j_2$.

2. *A company expects to receive \$4,000,000 in 2 years time, and pay out \$13,000,000 in 6 years time. If the current spot rates are as in the following table:*

   | *Term(years)* | 2 | 6 | 7 | 8 |
   |---|---|---|---|---|
   | *rate* | *3.5%* | *4.6%* | *4.8%* | *5%* |

   *(a) find a way for the company Reddington immunise these cash-flows by buying or selling zero-coupon bonds with maturities in 7 or 8 years.*

   Let the amounts purchased be $A_7$ and $A_8$. We want the cash-flows to have matching modified durations and present values. Matching present values gives:

   $$4000000(1.035)^{-2} + A_7(1.048)^{-7} + A_8(1.05)^{-8} = 13000000(1.046)^{-6}$$

while matching modified durations gives

$$\frac{4000000(1.035)^{-3} \times 2 + 7A_7(1.048)^{-8} + 8A_8(1.05)^{-9}}{4000000(1.035)^{-2} + A_7(1.048)^{-7} + A_8(1.05)^{-8}} = 6(1.046)^{-1}$$

Substituting the first equation into the second gives

$$4000000(1.035)^{-3}\times 2+7A_7(1.048)^{-8}+8A_8(1.05)^{-9} = 6\times 13000000(1.046)^{-7}$$

Subtracting 7 times the first equation from 1.048 times this equation gives

$$\begin{aligned}(8 \times 1.048 - 7(1.05))A_8(1.05)^{-9} - (7(1.035) - 2(1.048))4000000(1.035)^{-3}\\ = (6(1.048) - 7(1.046))13000000(1.046)^{-7}\end{aligned}$$

$$(1.034)A_8(1.05)^{-9}-(5.149)4000000(1.035)^{-3} = -(1.034)13000000(1.046)^{-7}$$

so

$$A_8 = \frac{(1.05)^9}{8(1.048) - 7(1.05)}(20596000(1.035)^{-3}-13442000(1.046)^{-7} = \$13,149,923.34$$

Substituting into the first equation gives

$$\begin{aligned}A_7 = (1.048)^7(13000000(1.046)^{-6} - 4000000(1.035)^{-2} - 13149923.34(1.05)^{-8})\\ = -\$3,761,173.83\end{aligned}$$

*(b) Is the immunisation in (a) a full immunisation?*

The immunisation in (a) is a full immunisation with respect to a parallel shift in the term structure, because the payments received are outside the payments made, so the modified duration of the payments received changes faster than the payments made.

3. *Assume a flat term structure of $j_2 = 4.6\%$. A company has issued a 20-year bond with face value \$120,000 and semi-annual coupon rate 6%. It plans to immunise these liabilites with two payments received in 3 and 17 years. Calculate these two payments.*

   The present value of the 20 year bond can be calculated using Makeham's formula. $K = 120000(1.023)^{-40} = \$48,323.22$, and $P = 48323.22 + (120000 - 48323.22)\frac{6}{4.6} = \$141,814.67$. The duration is given by $D = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((1+i)^n-1)+i} = \frac{1.023}{0.023} - \frac{1.023+40(0.007)}{0.03((1.023)^{40}-1)+0.023} = 25.174$ periods (or 12.59 years). If the payments received are $A_3$ and $A_{17}$, then the duration of these two payments must also be 12.59 years, so if the present values are $P_3$ and $P_{17}$ respectively, we must have $3P_3+17P_{17} = 12.59(P_3+P_{17})$. Let $x = \frac{P_3}{P_3+P_{17}}$, then $3x + 17(1 - x) = 12.59$, so $14x = 4.41$ or $x = 0.315211$. We also know that $P_3 + P_{17} = P = \$141,814.67$, so $P_3 = 0.315211 \times 141814.67 = \$44,701.59$ and $P_{17} = 141814.67 - 44701.59 = \$97,113.08$, so $A_3 = 44701.59(1.023)^6 = \$51,236.18$ and $A_{17} = 97113.08(1.023)^{34} = \$210,401.71$.

4. *The current term structure has the following yields on zero-coupon bonds:*

| *Term(years)* | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 |
|---|---|---|---|---|---|---|
| *rate $j_2$* | *2.1%* | *2.8%* | *3.5%* | *3.9%* | *4.3%* | *4.6%* |

*Calculate the modified duration of a 9% semi-annual 3-year bond, based on a parallel shift in the term structure.*

For a bond with face value \$100, the modified duration with respect to a shift in the half-year rate of interest is given by

$$\frac{4.5(1.0105)^{-2} + 2 \times 4.5(1.014)^{-3} + 3 \times 4.5(1.0175)^{-4} + 4 \times 4.5(1.0195)^{-5} + 5 \times 4.5(1.0215)^{-6} + 6 \times 104.5(1.023)^{-7}}{4.5(1.0105)^{-1} + 4.5(1.014)^{-2} + 4.5(1.0175)^{-3} + 4.5(1.0195)^{-4} + 4.5(1.0215)^{-5} + 104.5(1.023)^{-6}} = 5.30$$

The modified duration with respect to $j_2$ is therefore 2.65.