MATH 3030, Abstract Algebra
Winter 2012
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Sample Midterm Examination

This practice exam deliberately has more questions than the real midterm. Some of the theoretical questions are directly from the notes, and some are new, requiring a little thought. The questions from the notes are intended to provide a complete list of theorems from the course that you might be asked to prove.

Basic Questions

1. Give an example of a prime ideal which is not maximal.
2. Let $R = M_2(\mathbb{Z}_2)$, the ring of 2×2 matrices over $\mathbb{Z}_2$. What are the elements of the ideal generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$?
3. How many ring homomorphisms are there from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{90}$?
4. What is the dimension of $\mathbb{Q}(\sqrt{3}+\sqrt{2})$ as a vector space over $\mathbb{Q}(\sqrt{2})$?
5. Let α be a zero of $f(x) = x^2 - 2$ in $\mathrm{GF}(25)$. Find a generator of the multiplicative group of nonzero elements of $\mathrm{GF}(25)$.
6. Show that $x^3 + x + 1$ has distinct zeros in the algebraic closure of $\mathbb{Z}_5$.
7. Let α be a zero of $x^3 + x^2 + 2$ over $\mathbb{Z}_3$. Find $\mathrm{Irr}(\alpha+1, \mathbb{Z}_3)$.
8. Show that the set of polynomials $\{f \in \mathbb{Z}[x] | f(0) \text{ is divisible by } 3\}$ is an ideal in $\mathbb{Z}[x]$. Is it principal?
9. Compute a composition series for $D_5 \times D_4$. Is $D_5 \times D_4$ solvable?
10. Find a basis for $\mathbb{Q}(\sqrt{3}+\sqrt{5})$ over $\mathbb{Q}$.

Theoretical Questions

Results from Notes

11. Show that the composite of two ring homomorphisms is a ring homomorphism.
12. Prove that for a field F, every ideal in the polynomial ring $F[x]$ is principal.
13. Prove that if R is a commutative unital ring, and I is an ideal of R, then R/I is a field if and only if I is maximal.

14. Prove that if R is a commutative unital ring, and I is an ideal of R, then R/I is an integral domain if and only if I is prime.
15. Prove that given a field F, and a non-constant polynomial $f \in F[x]$, there is an extension field E of F containing a zero of f.
16. Prove that if E is a finite extension of F and K is a finite extension of E, then K is a finite extension of F and

$$[K : F] = [K : E][E : F]$$

17. Prove that the number of elements in a finite field is always a prime power.
18. Show that a subgroup of a solvable group is solvable.
19. Show that a field of characteristic $p \neq 0$ contains a subfield isomorphic to $\mathbb{Z}_p$.
20. Show that for an extension field E of F, and an element $\alpha \in E$, algebraic over F, there is an irreducible polynomial $p \in F[x]$ such that $p(\alpha) = 0$, and that this p is unique up to multiplication by a constant.
21. Show that any finite extension field E of a field F is algebraic over F.
22. (a) Show that if K is an algebraic extension of E, and E is an algebraic extension of F, then K is an algebraic extension of F.

    (b) Deduce that if M is a maximal algebraic extension of F (i.e. M is not a proper subfield of any other algebraic extension of F) then M is algebraically closed.
23. Show that it is not possible to construct a line segment of length $\sqrt[3]{2}$, starting from a line segment of length 1, and using only a straight-edge and compass.
24. Describe how to construct a finite field with p^n elements for a prime p as a subfield of $\overline{\mathbb{Z}_p}$, and explain what steps are needed to show that this is indeed a field.
25. State and prove the first isomorphism theorem.
26. State and prove the second isomorphism theorem.
27. State and prove the third isomorphism theorem.

New questions

28. Let E be an extension of F. Let $\alpha \in E$ be algebraic over F, and let $\beta \in E$ be transcendental over F. Must β be transcendental over $F(\alpha)$? Give a proof or a counterexample.

29. Let E be algebraically closed. Let F be a subfield of E. Show that the algebraic closure of F in E is algebraically closed.

30. Prove that the only irreducible polynomials in $\mathbb{R}[x]$ have degree less than 2. [You may assume the fundamental theorem of algebra — the complex numbers are algebraically closed.]

31. Show that no finite field is algebraically closed.

32. Show that for any n, and any prime p, there is an irreducible polynomial of degree n in $\mathbb{Z}_p$.

33. Show that any non-zero ring homomorphism between two fields is one-to-one.

34. Show that any algebraic extension of $\mathbb{R}$ is either $\mathbb{R}$, or else is isomorphic to $\mathbb{C}$. [Hint: the only irreducible polynomials in $\mathbb{R}$ are quadratic or linear.]

35. Show that the direct product of two solvable groups is solvable.

36. Show that the set of elements x satisfying $x^n = 0$ for some n is an ideal in any commutative ring R.

37. Let R be a commutative ring, and let $a \in R$. Show that the set $\{x \in R | ax = 0\}$ is an ideal in R.

38. Let α be a primitive root of unity in $\mathrm{GF}(p^n)$. Show that $\deg(\alpha, \mathbb{Z}_p) = n$.